

UNIFORM ESTIMATES FOR THE SEMI-PERIODIC EIGENVALUES OF THE SINGULAR DIFFERENTIAL OPERATORS

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ABSTRACT. Let $m \in \mathbb{N}$, $\alpha \in [0, 1]$, and V be a 1-periodic complex-valued distribution in the negative Sobolev space $H^{-m\alpha}[0, 1]$. The singular non-self-adjoint eigenvalue problem $D^{2m}u + Vu = \lambda u$, $D = -id/dx$, with semi-periodic boundary conditions is investigated. The uniform in V asymptotic and non-asymptotic eigenvalue estimates are found and proved. The case of periodic boundary conditions was earlier studied by authors in the papers [5, 6].

1. INTRODUCTION AND MAIN RESULTS

Consider the eigenvalue problem on the interval $[0, 1]$

$$D^{2m}u(x) + V(x)u(x) = \lambda u(x), \quad D = -id/dx$$

with semi-periodic boundary conditions. Here $V(x)$ is a 1-periodic complex-valued distribution in the negative Sobolev space $H^{-m\alpha}[0, 1]$ with

$$m \in \mathbb{N}, \quad \alpha \in [0, 1].$$

To investigate the problem we associate with one an unbounded linear operator L in an appropriate Hilbert space and after that we study a spectrum of the operator L .

If $V(x)$ belongs to the Hilbert space $L^2[0, 1]$ then the differential expression

$$l[\cdot] := D^{2m} + V(x)$$

is regular and semi-periodic boundary conditions

$$u^{(k)}(0) = -u^{(k)}(1), \quad k \in \{0, 1, \dots, 2m-1\}$$

are regular in the Birkhoff sense. In this case there exists the unbounded linear operator L in the Hilbert space $L^2[0, 1]$ with the dense domain

$$Dom(L) = \left\{ u \in H^{2m}[0, 1] \mid u^{(k)}(0) = -u^{(k)}(1), k = 0, 1, \dots, 2m-1 \right\}$$

such that

$$Lu = l[u], \quad u \in Dom(L).$$

The spectrum $spec(L)$ of L is discrete and consists of a sequence of eigenvalues $\{\lambda_k\}_{k \geq 1}$ with the property $Re\lambda_n \rightarrow \infty$ for $n \rightarrow \infty$, where the eigenvalues λ_n are enumerated with there algebraic multiplicities and ordered lexicographically so that

$$Re\lambda_k < Re\lambda_{k+1}, \quad \text{or} \quad Re\lambda_k = Re\lambda_{k+1} \quad \text{and} \quad Im\lambda_k \leq Im\lambda_{k+1}.$$

An asymptotic behaviour of the eigenvalues of L in this case was investigated earlier in detail (see [8] and references therein). It has the following form:

$$\lambda_{2n-1}, \lambda_{2n} = (2n-1)^{2m} \pi^{2m} + O(n^{2m-3/2}), \quad n \rightarrow \infty,$$

since semi-periodic boundary conditions are not strongly regular [8]. This general asymptotic formula contains two power terms. We will prove below that in this situation

$$\lambda_{2n-1}, \lambda_{2n} = (2n-1)^{2m} \pi^{2m} + \widehat{V}(0) \pm \sqrt{\widehat{V}(-2(2n-1)) \widehat{V}(2(2n-1))} + o(n^{-m/2}), \quad n \rightarrow \infty,$$

where $\widehat{V}(k)$ denote the Fourier coefficients of $V(x)$. The last formula contains $2m+1$ power terms and in general non-power term

$$\pm \sqrt{\widehat{V}(-2(2n-1)) \widehat{V}(2(2n-1))} \in l^2(\mathbb{N}).$$

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The aim of this article is to study the semi-periodic eigenvalue problem in a singular case when $V(x)$ is a 1-periodic complex-valued distribution in the negative Sobolev space $H^{-m\alpha}[0, 1]$. To do it we will consider the problem in the negative Sobolev space $H_-^{-m}[0, 1]$ of semi-periodic distributions. Then the operator $L \equiv L_m(V)$ has the natural domain

$$\text{Dom}(L) = H_-^m[0, 1].$$

Here we use the following notation. The complex Sobolev spaces $H_+^s[0, 1]$, $s \in \mathbb{R}$, of 1-periodic functions or distributions are defined by means their Fourier coefficients:

$$H_+^s[0, 1] := \left\{ f = \sum_{k \in \mathbb{Z}} \widehat{f}(2k) e^{i2k\pi x} \mid \|f\|_{H_+^s[0, 1]} < \infty \right\},$$

where

$$\|f\|_{H_+^s[0, 1]} := \left(\sum_{k \in \mathbb{Z}} \langle 2k \rangle^{2s} |\widehat{f}(2k)|^2 \right)^{1/2}, \quad \langle k \rangle := 1 + |k|,$$

$$\widehat{f}(2k) := \langle f, e^{i2k\pi x} \rangle, \quad k \in \mathbb{Z}.$$

The brackets denote the sesquilinear pairing between dual spaces $H_+^s[0, 1]$ and $H_+^{-s}[0, 1]$ extending the $L^2[0, 1]$ -inner product

$$\langle f, g \rangle := \int_0^1 f(x) \overline{g(x)} dx, \quad f, g \in L^2[0, 1].$$

In the same fashion the complex Sobolev spaces $H_-^s[0, 1]$, $s \in \mathbb{R}$, of semi-periodic functions or distributions are introduced:

$$H_-^s[0, 1] := \left\{ f = \sum_{k \in \mathbb{Z}} \widehat{f}(2k+1) e^{i(2k+1)\pi x} \mid \|f\|_{H_-^s[0, 1]} < \infty \right\},$$

where

$$\|f\|_{H_-^s[0, 1]} := \left(\sum_{k \in \mathbb{Z}} \langle 2k+1 \rangle^{2s} |\widehat{f}(2k+1)|^2 \right)^{1/2}, \quad \langle k \rangle := 1 + |k|,$$

$$\widehat{f}(2k+1) := \langle f, e^{i(2k+1)\pi x} \rangle, \quad k \in \mathbb{Z}.$$

Here the brackets denote the sesquilinear pairing between dual spaces $H_-^s[0, 1]$ and $H_-^{-s}[0, 1]$ extending the $L^2[0, 1]$ -inner product.

Obviously that

$$H_+^0[0, 1] = H_-^0[0, 1] = L^2[0, 1].$$

Also we use the weighted l^2 -spaces

$$h^{s,n} \equiv h^{s,n}(\mathbb{Z}; \mathbb{C})$$

for any $n \in \mathbb{Z}$ and $s \in \mathbb{R}$. These spaces are the Hilbert spaces of sequences $(a(k))_{k \in \mathbb{Z}}$ in \mathbb{C} with the norm

$$\|a\|_{h^{s,n}} := \left(\sum_{k \in \mathbb{Z}} \langle k+n \rangle^{2s} |a(k)|^2 \right)^{1/2}.$$

For $n=0$ we will simply write h^s instead of $h^{s,0}$. To shorten notation, it is convenient to denote by $h^s(n)$ the n -th element of a sequence $(a(k))_{k \in \mathbb{Z}}$ in h^s . It is clear that if $a \in h^s$, then

$$a(n) = o(|n|^{-s}), \quad |n| \rightarrow \infty.$$

The following Theorem summarizes the main results of this paper.

Theorem 1.1. *Let $V \in H_+^{-m\alpha}[0, 1]$, $m \in \mathbb{N}$, $\alpha \in [0, 1]$, and $R > 0$.*

(1) *Let $\alpha = 1$.*

(a) *There exists $\varepsilon > 0$ such that for any $W \in H_+^{-m}[0, 1]$ with*

$$\|W - V\|_{H_+^{-m}[0, 1]} \leq \varepsilon$$

the eigenvalues of the operator $L_m(W)$ satisfy the asymptotic formulae

$$\lambda_{2n-1}(m, W), \lambda_{2n}(m, W) = (2n-1)^{2m} \pi^{2m} + O(n^m), \quad n \rightarrow \infty$$

uniformly in W .

(b) For any $W \in H_+^{-m}[0, 1]$ with

$$\|W - V\|_{H_+^m[0,1]} \leq R$$

the eigenvalues of the operator $L_m(W)$ satisfy the asymptotic formulae

$$\lambda_{2n-1}(m, W), \lambda_{2n}(m, W) = (2n-1)^{2m} \pi^{2m} + o(n^m), \quad n \rightarrow \infty$$

uniformly in W .

(2) Let $\alpha \in [1/2, 1)$. For any $V \in H_+^{-m\alpha}[0, 1]$ with

$$\|V\|_{H_+^{-m\alpha}[0,1]} \leq R$$

the eigenvalues of the operator $L_m(V)$ satisfy the asymptotic formulae

$$\begin{aligned} \lambda_{2n-1}(m, V), \lambda_{2n}(m, V) &= (2n-1)^{2m} \pi^{2m} + \widehat{V}(0) \\ &\pm \sqrt{\widehat{V}(-2(2n-1)) \widehat{V}(2(2n-1))} + h^{m(1-2\alpha)-\varepsilon}(n), \end{aligned}$$

uniformly in V .

(3) Let $\alpha \in [0, 1/2)$. For any $V \in H_+^{-m\alpha}[0, 1]$ with

$$\|V\|_{H_+^{-m\alpha}[0,1]} \leq R$$

the eigenvalues of the operator $L_m(V)$ satisfy the asymptotic formulae

$$\begin{aligned} \lambda_{2n-1}(m, V), \lambda_{2n}(m, V) &= (2n-1)^{2m} \pi^{2m} + \widehat{V}(0) \\ &\pm \sqrt{\widehat{V}(-2(2n-1)) \widehat{V}(2(2n-1))} + h^{m(1/2-\alpha)}(n), \end{aligned}$$

uniformly in V .

If the distribution $V(x)$ is *real-valued* and $\alpha \in [0, 1/2)$ then our asymptotic formulae are of the form

$$\begin{aligned} \lambda_{2n-1}(m, V) &= (2n-1)^{2m} \pi^{2m} + \widehat{V}(0) - |\widehat{V}(4n-2)| + h^{m(1/2-\alpha)}(n), \\ \lambda_{2n}(m, V) &= (2n-1)^{2m} \pi^{2m} + \widehat{V}(0) + |\widehat{V}(4n-2)| + h^{m(1/2-\alpha)}(n) \end{aligned}$$

and in the case $m = 1, \alpha = 0$ reproduce the Marchenko's estimates [4]. They turn out to be uniform on bounded sets of $V \in L^2[0, 1]$.

An improved version of the eigenvalue estimates for $m \in \mathbb{N}, \alpha \in [0, 1/2)$ is given in Section 5. The remainder terms in these estimates are in $h^{m(1-2\alpha)-\varepsilon}, \varepsilon > 0$.

Also we prove in Section 3 and Section 6 the following non-asymptotic estimates for the eigenvalues.

Theorem 1.2. Let $V \in H_+^{-m\alpha}[0, 1], m \in \mathbb{N}, \alpha \in [0, 1], C > 1$ and $R > 0$.

(1) Let $\alpha = 1$. There exist $\varepsilon > 0, M \geq 1$ and $n_0 \in \mathbb{N}$ such that for any $W \in H_+^{-m}[0, 1]$ with

$$\|W - V\|_{H_+^{-m}[0,1]} \leq \varepsilon$$

for all $n > n_0$ the estimates:

$$\begin{aligned} |\lambda_{2n-1}(m, W) - (2n-1)^{2m} \pi^{2m}| &< (2n-1)^m, \\ |\lambda_{2n}(m, W) - (2n-1)^{2m} \pi^{2m}| &< (2n-1)^m. \end{aligned}$$

are hold.

(2) Let $\alpha \in [0, 1)$. There exist $M = M(R) \geq 1$ and $n_0 = n_0(R, C) \in \mathbb{N}$ such that for any $V \in H_+^{-m\alpha}[0, 1]$ with

$$\|V\|_{H_+^{-m\alpha}[0,1]} \leq R$$

for all $n > n_0$ the estimates:

$$\begin{aligned} |\lambda_{2n-1}(m, V) - (2n-1)^{2m} \pi^{2m}| &< 3^m \sqrt{2} C R (2n-1)^{m\alpha}, \\ |\lambda_{2n}(m, V) - (2n-1)^{2m} \pi^{2m}| &< 3^m \sqrt{2} C R (2n-1)^{m\alpha}. \end{aligned}$$

are hold. The constant n_0 is efficient.

The similar results for the periodic eigenvalues were proved in the papers [2, 7] ($m = 1$) and [5, 6] ($m \geq 1$).

2. THE SPECTRAL PROBLEM IN THE HILBERT SEQUENCE SPACE

In this Section we introduce and study the matrix operator T in the Hilbert sequence space which is unitary equivalent to the differential operator L and has the same spectrum.

Further we denote by $h_+^{s,n} \equiv h_+^{s,n}(\mathbb{Z}; \mathbb{C})$ and $h_-^{s,n} \equiv h_-^{s,n}(\mathbb{Z}; \mathbb{C})$ the subspaces of $h^{s,n}(\mathbb{Z}; \mathbb{C})$ defined by

$$\begin{aligned} h_+^{s,n} &:= \{a \in h^{s,n} | a(2k+1) = 0, \forall k \in \mathbb{Z}\}, \\ h_-^{s,n} &:= \{a \in h^{s,n} | a(2k) = 0, \forall k \in \mathbb{Z}\}. \end{aligned}$$

And also we denote by

$$h_{+,0}^{s,n} \equiv h_{+,0}^{s,n}(\mathbb{Z}; \mathbb{C})$$

the subspace of $h_+^{s,n}(\mathbb{Z}; \mathbb{C})$ defined by

$$h_{+,0}^{s,n} := \{a \in h_+^{s,n} | a(0) = 0\}.$$

Obviously that

$$h^{s,n} = h_+^{s,n} \oplus h_-^{s,n}, \quad s \in \mathbb{R}, n \in \mathbb{Z}.$$

The map

$$f \mapsto (\hat{f}(2k))_{k \in \mathbb{Z}}$$

is an isometric isomorphism of the space $H_+^s[0, 1]$ onto h_+^s , and the map

$$g \mapsto (\hat{g}(2k+1))_{k \in \mathbb{Z}}$$

is an isometric isomorphism of the space $H_-^s[0, 1]$ onto h_-^s , $s \in \mathbb{R}$.

For these isomorphisms the multiplication of functions corresponds to convolution of sequences, where the convolution product of two sequences

$$a = (a(k))_{k \in \mathbb{Z}}, \quad b = (b(k))_{k \in \mathbb{Z}}$$

(formally) defined as the sequence given by

$$(2.1) \quad (a * b)(k) := \sum_{j \in \mathbb{Z}} a(k-j)b(j).$$

So, given two functions f, g formally,

$$(2.2) \quad (\widehat{f \cdot g})(k) = \sum_{j \in \mathbb{Z}} \hat{f}(k-j)\hat{g}(j).$$

The following Convolution Lemma is the modification of the Main Convolution Lemma [2] and very important for our method.

Lemma 2.1 (Convolution Lemma). *Let $n \in \mathbb{Z}$, $s, r \geq 0$, and $t \in \mathbb{R}$ with $t \leq \min(s, r)$. If $s + r - t > 1/2$, than the convolution map is continuous (uniformly in n), when viewed as a map*

$$\begin{aligned} (a') \quad h_+^{r,n} \times h_-^{s,-n} &\longrightarrow h_-^t, & (a'') \quad h_+^{r,n} \times h_+^{s,-n} &\longrightarrow h_+^t, & (a''') \quad h_-^{r,n} \times h_-^{s,-n} &\longrightarrow h_+^t, \\ (b') \quad h_+^{-t} \times h_-^{s,n} &\longrightarrow h_-^{-r,n}, & (b'') \quad h_+^{-t} \times h_+^{s,n} &\longrightarrow h_+^{-r,n}, & (b''') \quad h_-^{-t} \times h_-^{s,n} &\longrightarrow h_+^{-r,n}, \\ (c') \quad h_+^t \times h_-^{-s,n} &\longrightarrow h_-^{-r,n}, & (c'') \quad h_+^t \times h_+^{-s,n} &\longrightarrow h_+^{-r,n}, & (c''') \quad h_-^t \times h_-^{-s,n} &\longrightarrow h_+^{-r,n}. \end{aligned}$$

So, the maps

$$(2.3) \quad H_+^{-m\alpha}[0, 1] \times H_-^{m(2-\alpha)}[0, 1] \mapsto H_-^{-m\alpha}[0, 1], \quad (V, f) \mapsto V \cdot f,$$

$$(2.4) \quad H_+^{m(2-\alpha)}[0, 1] \times H_-^{-m\alpha}[0, 1] \mapsto H_-^{-m\alpha}[0, 1], \quad (V, f) \mapsto V \cdot f$$

are continuous, when $V \cdot f$ is given by formula (2.2).

For a distribution

$$f = \sum_{k \in \mathbb{Z}} \hat{f}(2k)e^{i2k\pi x}$$

we can define the conjugate distribution

$$\bar{f} = \sum_{k \in \mathbb{Z}} \overline{\hat{f}(-2k)}e^{i2k\pi x}.$$

A distribution $f \in H_+^s[0, 1]$ is said to be real-valued if $\bar{f} = f$, i.e. the corresponding sequence of the Fourier coefficients is Hermitian-symmetric:

$$\overline{\hat{f}(2k)} = \hat{f}(-2k), \quad \forall k \in \mathbb{Z}.$$

Let $m \in \mathbb{N}$, $\alpha \in [0, 1]$, and v be in $h_+^{-m\alpha}$. Consider in the Hilbert sequence space $h_-^{-m\alpha}$ the unbounded linear operator

$$T_\alpha \equiv T_\alpha(v) := A^m + B(v), \quad T_1 \equiv T$$

with the dense domain

$$\text{Dom}(T_\alpha) = h_-^{m(2-\alpha)},$$

where A^m and $B(v)$ are the infinite matrices,

$$\begin{aligned} A(2k-1, 2j-1) &:= (2k-1)^2 \pi^2 \delta_{kj}, & A(2k, 2j) &:= 0, \\ A^m(2k-1, 2j-1) &:= (2k-1)^{2m} \pi^{2m} \delta_{kj}, & A^m(2k, 2j) &:= 0, \quad k, j \in \mathbb{Z} \end{aligned}$$

and

$$B(v)(2k-1, 2j-1) := v(2k-2j), \quad B(v)(2k, 2j) := 0, \quad k, j \in \mathbb{Z}.$$

Obviously, that the operator A^m is a positive self-adjoint operator in the Hilbert sequence space $h_-^{-m\alpha}$ with the dense domain

$$\text{Dom}(A^m) = h_-^{m(2-\alpha)}.$$

The spectrum of A^m is discrete:

$$\text{spec}(A^m) = \{(2k-1)^{2m} \pi^{2m} \mid k \in \mathbb{N}\},$$

where all eigenvalues are double.

Lemma 2.2. *The operator $B(v)$, $v \in h_+^{-m\alpha}$ with the domain*

$$\text{Dom}(B(v)) = h_-^{m(2-\alpha)}$$

is A^m -bounded, and its relative bound is equal 0.

Proof. According to the Convolution Lemma there exist the constants $C_{\alpha,m}^{(1)} > 0$ and $C_{\alpha,m}^{(2)} > 0$ such that

$$\|B(v)u\|_{h_-^{-m\alpha}} \leq \begin{cases} C_{\alpha,m}^{(1)} \|v\|_{h_+^{m(2-\alpha)}} \|u\|_{h_-^{-m\alpha}}, & v \in h_+^{m(2-\alpha)}, u \in h_-^{-m\alpha}, \\ C_{\alpha,m}^{(2)} \|v\|_{h_+^{-m\alpha}} \|u\|_{h_-^{m(2-\alpha)}}, & v \in h_+^{-m\alpha}, u \in h_-^{m(2-\alpha)}. \end{cases}$$

Further, for any fixed $\delta > 0$ there exists a decomposition

$$v = v_0 + v_\delta$$

with

$$v_0 \in h_+^{m(2-\alpha)}, \quad v_\delta \in h_+^{-m\alpha}, \quad \|v_\delta\|_{h_+^{-m\alpha}} < \frac{\delta}{C_{\alpha,m}^{(2)}}.$$

Taking into account that

$$\|u\|_{h_-^{m(2-\alpha)}} \leq \|u\|_{h_-^{-m\alpha}} + \|A^m u\|_{h_-^{-m\alpha}}, \quad u \in h_-^{m(2-\alpha)}$$

then we have the following estimates:

$$\begin{aligned} \|B(v)u\|_{h_-^{-m\alpha}} &\leq \|B(v_0)u\|_{h_-^{-m\alpha}} + \|B(v_\delta)u\|_{h_-^{-m\alpha}} \\ &\leq C_{\alpha,m}^{(1)} \|v_0\|_{h_+^{m(2-\alpha)}} \|u\|_{h_-^{-m\alpha}} + C_{\alpha,m}^{(2)} \|v_\delta\|_{h_+^{-m\alpha}} \|u\|_{h_-^{m(2-\alpha)}} \\ &\leq \delta \|A^m u\|_{h_-^{-m\alpha}} + \left(C_{\alpha,m}^{(2)} \|v_0\|_{h_+^{m(2-\alpha)}} + \delta \right) \|u\|_{h_-^{-m\alpha}}. \end{aligned}$$

Hence $B(v) \ll A^m$. □

Corollary 2.3. *The operator $B(v)$ is form-bounded with respect to operator A^m and its relative bound is equal 0.*

Corollary 2.4. *The operator T_α is quasi-sectorial. More precisely, for any $\varepsilon > 0$ there exists $c_\varepsilon > 0$ such that for any $f \in \text{Dom}(T_\alpha)$*

$$\left| \arg((T_\alpha + c_\varepsilon Id)f, f)_{h_-^{-m\alpha}} \right| \leq \varepsilon.$$

Proposition 2.5. *Let $m \in \mathbb{N}$, $\alpha \in [0, 1]$, and v be in $h_+^{-m\alpha}$.*

- (1) *The operator T_α is quasi- m -sectorial.*
- (3) *A resolvent set of the operator T_α is not empty and its resolvent $R(\lambda, T_\alpha)$ is a compact operator.*

Proof. Let prove the Proposition.

- (1) The operator T_α is quasi-sectorial. Its maximality property follows from Statement (3) of this Proposition.
- (3) For any $\lambda \in \mathbb{C}$ with $\operatorname{Re} \lambda \leq 0$ the following estimates are valid:

$$\|A^m R(\lambda, A^m)\|_{\mathcal{L}(h_-^{m\alpha})} \leq \frac{2^{2m} \pi^{2m}}{|\lambda - \pi^{2m}|},$$

$$\|R(\lambda, A^m)\|_{\mathcal{L}(h_-^{m\alpha})} \leq \frac{1}{|\lambda - \pi^{2m}|}.$$

Therefore the formulated statement follows from Theorem 3.17 ([3], Ch. IV) and the inequalities since the resolvent $R(\lambda, A^m)$ is a compact operator. \square

Corollary 2.6. *The spectrum $\operatorname{spec}(T_\alpha)$ of the operator T_α is discrete and consists of a sequence of the eigenvalues*

$$\lambda_k = \lambda_k(\alpha, m, v), \quad k \in \mathbb{N}$$

with the property that

$$\operatorname{Re} \lambda_k \rightarrow +\infty \quad \text{as } k \rightarrow +\infty,$$

where the eigenvalues λ_k are enumerated with their algebraic multiplicities ordered lexicographically.

If $v \in h_+^{-m\alpha}$, then for any β such that

$$0 \leq \alpha \leq \beta \leq 1,$$

the operator T_β is well defined and its spectrum is discrete.

Proposition 2.7. *If α and β as above, then*

$$\operatorname{spec}(T_\beta) = \operatorname{spec}(T_\alpha).$$

Proof. Obviously that

$$\operatorname{spec}(T_\beta) \supseteq \operatorname{spec}(T_\alpha)$$

since

$$T_\beta \supseteq T_\alpha.$$

To prove the converse inclusion for the spectra sufficient to show that any eigenvector (or root vector) $f \in \operatorname{Dom}(T_\beta) = h_-^{m\beta}$ of T_β is in fact an element of $\operatorname{Dom}(T_\alpha) = h_-^{m\alpha}$. So, let $\lambda \in \operatorname{spec}(T_\beta)$, and

$$(T_\beta - \lambda \operatorname{Id})f = g, \quad f, g \in \operatorname{Dom}(T_\beta),$$

where f is eigenvector if $g = 0$, and root vector if $g \neq 0$. Taking into account that

$$B(v)f \in h_-^{m\alpha}$$

by the Convolution Lemma, we conclude that

$$A^m f = g + \lambda f - B(v)f \in h_-^{m\alpha}.$$

Since

$$(A^m)^{-1} \in \mathcal{L}(h_-^{m\alpha}, h_-^{m(2-\alpha)}),$$

we have got

$$f = (A^m)^{-1} A^m f \in h_-^{m(2-\alpha)}$$

as claimed. This establish that

$$\operatorname{spec}(T_\beta) = \operatorname{spec}(T_\alpha),$$

and the proof is complete. \square

To study the eigenvalue problem

$$Tu = \lambda u$$

we will compare the spectrum $\operatorname{spec}(T_\alpha)$ of the operator

$$T_\alpha = A^m + B(v)$$

with the spectrum of the unperturbed operator A^m in the same space,

$$\operatorname{spec}(A^m) = \{(2k-1)^{2m} \pi^{2m} \mid k \in \mathbb{N}\}.$$

And take into account that for any $\alpha \in [0, 1]$ the operator T is isospectral to the operator T_α .

Further, for given $M \geq 1$, $n \geq 1$, and $0 < r_n < (2n-1)^m \pi^{2m}$ the following regions Ext_M and $Vert_n^m(r_n)$ of complex plane will be used:

$$Ext_M := \{\lambda \in \mathbb{C} \mid Re \lambda \leq |Im \lambda| - M\},$$

$$Vert_n^m(r_n) := \{\lambda = (2n-1)^{2m} \pi^{2m} + z \in \mathbb{C} \mid |Re z| \leq (2n-1)^m \pi^{2m}, |z| \geq r_n\}.$$

Adding a constant to a convolution operator $B(v)$ results in a shift of the spectrum of the operator

$$T_\alpha = A^m + B(v)$$

by the same constant. Therefore we assume bellow, without loss of generality, that

$$v(0) = 0.$$

It is clear that

$$|v(0)| \leq \|v\|_{h_+^{-m\alpha}}.$$

3. NON-ASYMPTOTIC ESTIMATES

In this Section we will prove some non-asymptotic estimates for the eigenvalues of $spec(T) = spec(T_\alpha)$. For this purpose let decompose the operator

$$\lambda - A^m - B(v)$$

in the following way. For $\lambda \in \mathbb{C} \setminus spec(A^m)$ write

$$\lambda - A^m - B(v) = A_\lambda^{m/2} (I_\lambda - S_\lambda) A_\lambda^{m/2},$$

where $A_\lambda^{m/2}$, I_λ and S_λ are the following infinite matrices ($k, j \in \mathbb{Z}$)

$$\begin{aligned} A_\lambda^m(2k-1, 2j-1) &:= |\lambda - (2k-1)^{2m} \pi^{2m}| \delta_{kj}, & A_\lambda^m(2k, 2j) &= 0, \\ I_\lambda(2k-1, 2j-1) &:= \frac{\lambda - (2k-1)^{2m} \pi^{2m}}{|\lambda - (2k-1)^{2m} \pi^{2m}|} \delta_{kj}, & I_\lambda(2k, 2j) &= 0, \\ S_\lambda(2k-1, 2j-1) &:= \frac{v(2k-2j)}{|\lambda - (2k-1)^{2m} \pi^{2m}|^{1/2} |\lambda - (2j-1)^{2m} \pi^{2m}|^{1/2}}, & S_\lambda(2k, 2j) &= 0. \end{aligned}$$

Note that $A_\lambda^{m/2}$ and I_λ are diagonal matrices independent on v . Both I_λ and S_λ can be viewed as linear operators on h_-^0 . The reason for working with I_λ instead of the identity matrix Id is that we want to avoid having to take complex square roots in the definitions of $A_\lambda^{m/2}$ and S_λ , [7]. Clearly, for any $t, s \in \mathbb{R}$ with $s-t \leq 1$, and any $\lambda \in \mathbb{C} \setminus spec(A^m)$, we have $A_\lambda^{-m/2} \in \mathcal{L}(h_-^{mt}, h_-^{ms})$ with norm

$$(3.1) \quad \|A_\lambda^{-m/2}\|_{\mathcal{L}(h_-^{mt}, h_-^{ms})} = \sup_{k \in \mathbb{Z}} \frac{\langle 2k-1 \rangle^{m(s-t)}}{|\lambda - (2k-1)^{2m} \pi^{2m}|^{1/2}} < \infty.$$

Further, it is clearly that any $\lambda \in \mathbb{C} \setminus spec(A^m)$ with $\|S_\lambda\|_{\mathcal{L}(h_-^0)} < 1$ is in the resolvent set $Resol(T_\alpha)$ of

$$T_\alpha = A^m + B(v),$$

and the resolvent operator is of the form

$$(3.2) \quad (\lambda - A^m - B(v))^{-1} = A_\lambda^{-m/2} (I_\lambda - S_\lambda)^{-1} A_\lambda^{-m/2},$$

where the right side of (3.2) is viewed as a composition

$$h^{-m\alpha} \rightarrow h^0 \rightarrow h^0 \rightarrow h^m (\hookrightarrow h^{-m\alpha}).$$

In a straightforward way one can prove

Lemma 3.1. *Let $m \in \mathbb{N}$, $\alpha \in [0, 1]$, $M \geq 1$, and $v \in h_{+,0}^{-m\alpha}$. Then, for any $\lambda \in Ext_M$,*

$$\|S_\lambda\|_{\mathcal{L}(h_-^0)} \leq 2^{2m+1} \|v\|_{h_+^{-m\alpha}} \frac{1}{M^{(1-\alpha)/2+1/4}}.$$

Proof. We estimate the $\mathcal{L}(h_-^0)$ -norm of S_λ by its Hilbert-Schmidt norm,

$$(3.3) \quad \|S_\lambda\|_{\mathcal{L}(h_-^0)} \leq \left(\sum_{k,j} \frac{|v(2k-2j)|^2}{|\lambda - (2k-1)^{2m}\pi^{2m}||\lambda - (2j-1)^{2m}\pi^{2m}|} \right)^{1/2}.$$

Using

$$\langle k-j \rangle^{2m\alpha} \leq 2^{2m\alpha} (\langle k \rangle^{2m\alpha} + \langle j \rangle^{2m\alpha}), \quad (k, j \in \mathbb{Z})$$

together with the trigonometric estimate

$$|\lambda - k^{2m}\pi^{2m}| \geq (M + k^{2m}\pi^{2m}) \sin\left(\frac{\pi}{4}\right), \quad (k \in \mathbb{Z}, \lambda \in Ext_M),$$

one concludes

$$\begin{aligned} \|S_\lambda\|_{\mathcal{L}(h_-^0)} &\leq \left(\sum_{k,j} \frac{2 \cdot 2^{2m\alpha} (\langle 2k-1 \rangle^{2m\alpha} + \langle 2j-1 \rangle^{2m\alpha})}{(M + (2k-1)^{2m}\pi^{2m})(M + (2j-1)^{2m}\pi^{2m})} \langle 2k-2j \rangle^{-2m\alpha} |v(2k-2j)|^2 \right)^{1/2} \\ &\leq \left(4 \cdot 2^{2m\alpha} \sup_k \frac{\langle 2k-1 \rangle^{2m\alpha}}{(M + (2k-1)^{2m}\pi^{2m})} \sum_k \frac{1}{(M + (2k-1)^{2m}\pi^{2m})} \right)^{1/2} \\ &\quad \cdot \left(\sum_j \langle 2k-2j \rangle^{-2m\alpha} |v(2k-2j)|^2 \right)^{1/2} \\ &= 2^{m\alpha+1} \left(\sup_k \frac{\langle 2k-1 \rangle^{2m\alpha}}{(M + (2k-1)^{2m}\pi^{2m})} \right)^{1/2} \left(\sum_k \frac{1}{M + (2k-1)^{2m}\pi^{2m}} \right)^{1/2} \|v\|_{h_+^{-m\alpha}}. \end{aligned}$$

□

For $\lambda \in Vert_n^m(r_n)$, the following estimate for $\|S_\lambda\|_{\mathcal{L}(h_-^0)}$ can be obtained:

Lemma 3.2. *Let $m \in \mathbb{N}$, $\alpha \in [0, 1]$, $n \geq \frac{8m^2+4m-7}{2(8m-7)}$, $0 < r_n < (2n-1)^m\pi^{2m}$, and $v \in h_{+,0}^{-m\alpha}$. Then, for any $\lambda \in Vert_n^m(r_n)$,*

$$\begin{aligned} \|S_\lambda\|_{\mathcal{L}(h_-^0)} &\leq \frac{1}{r_n} (|v(2(2n-1))| + |v(-2(2n-1))|) \\ &\quad + 4 \left(\frac{2}{\pi} \right)^m \left(\frac{(2n-1)^{m(\alpha-1+1/2m)}}{\sqrt{r_n}} + \frac{6 \log(2n-1)}{(2n-1)^{m(1-\alpha)}} \right) \|v\|_{h_+^{-m\alpha}}. \end{aligned}$$

To prove Lemma 3.2, one uses that for $\lambda \in Vert_n^m(r_n)$, $n \geq \frac{8m^2+4m-7}{2(8m-7)}$, $k \neq \pm(2n-1)$

$$(3.4) \quad \frac{1}{|\lambda - k^{2m}\pi^{2m}|} \leq \frac{3}{\pi^{2m}} \frac{1}{|k^{2m} - (2n-1)^{2m}|},$$

together with the following elementary estimates:

Lemma 3.3. *Let $m \in \mathbb{N}$, $\alpha \in [0, 1]$, and $n \geq m$. Then*

- (a) $\sup_{k \neq \pm n} \frac{\langle k \rangle^{m\alpha}}{|k^{2m} - n^{2m}|^{1/2}} \leq 3^{m\alpha} n^{m(\alpha-1+\frac{1}{2m})};$
- (b) $\sup_{k \neq \pm n} \frac{\langle k \pm n \rangle^{m\alpha}}{|k^{2m} - n^{2m}|^{1/2}} \leq 4^{m\alpha} n^{m(\alpha-1+\frac{1}{2m})};$
- (c) $\sum_{k \neq \pm n} \frac{1}{|k^{2m} - n^{2m}|^{1/2}} \leq 5 \frac{1+\log n}{n}.$

Lemma 3.2 together with the estimate

$$(|v(2(2n-1))| + |v(-2(2n-1))|) \leq 3^m \sqrt{2} \|v\|_{h_+^{-m\alpha}} (2n-1)^{m\alpha}$$

leads to

$$\|S_\lambda\|_{\mathcal{L}(h_-^0)} \leq \frac{3^m \sqrt{2} (2n-1)^{m\alpha} \|v\|_{h_+^{-m\alpha}}}{r_n} + 4 \left(\frac{2}{\pi} \right)^m \left(\frac{(2n-1)^{m(\alpha-1+1/2m)}}{\sqrt{r_n}} + \frac{6 \log(2n-1)}{(2n-1)^{m(1-\alpha)}} \right) \|v\|_{h_+^{-m\alpha}}.$$

Combining this with Lemma 3.1 one obtains

Proposition 3.4. *Let $m \in \mathbb{N}$, $\alpha \in [0, 1)$, $R > 0$, $C > 1$, and $r_n := 3^m \sqrt{2}CR(2n-1)^{m\alpha}$ ($n \geq 1$). Then there exist $M = M(R) \geq 1$ and $n_0 = n_0(R, C) \geq \frac{8m^2+4m-7}{2(8m-7)}$ with $0 < r_n < (2n-1)^m \pi^{2m}$ ($n \geq n_0$) so that, for any $v \in h_{+,0}^{-m\alpha}$ with $\|v\|_{h_+^{-m\alpha}} \leq R$*

$$\|S_\lambda\|_{\mathcal{L}(h_-)} < 1 \quad \text{for } \lambda \in \text{Ext}_M \cup \bigcup_{n \geq n_0} \text{Vert}_n^m(r_n).$$

Hence

$$\text{Ext}_M \cup \bigcup_{n \geq n_0} \text{Vert}_n^m(r_n) \subseteq \text{Resol}(T_\alpha).$$

So, the spectrum $\text{spec}(T_\alpha(v))$ of $T_\alpha(v)$ is contained in the complement of the set

$$\text{Ext}_M \cup \bigcup_{n \geq n_0} \text{Vert}_n^m(r_n).$$

To localize the eigenvalues notice that for any $0 \leq s \leq 1$ in fact

$$\text{Ext}_M \cup \bigcup_{n \geq n_0} \text{Vert}_n^m(r_n) \subseteq \text{Resol}(T_\alpha(sv)).$$

Hence, for any contour

$$\Gamma \subseteq \text{Ext}_M \cup \bigcup_{n \geq n_0} \text{Vert}_n^m(r_n)$$

and any $0 \leq s \leq 1$, the Riesz projector

$$P(s) := \frac{1}{2\pi i} \int_\Gamma (\lambda - A^m - B(sv))^{-1} d\lambda \in \mathcal{L}(h^{-m\alpha})$$

is well defined and depends continuously on s . Since projectors whose difference has small norm have isomorphic ranges (see, e.g. [1, 3]), continuity of the map

$$s \mapsto P(s)$$

implies that the dimension of the range $P(s)$ is independent of s . Therefore the number of eigenvalues of $A^m + B(v)$ and A^m inside Γ (counted with their algebraic multiplicities) are the same.

Summarizing the result above, we obtain the following statement.

Theorem 3.5. *Let $m \in \mathbb{N}$, $\alpha \in [0, 1)$, $C > 1$, and $R > 0$. Then there exist $M = M(R) \geq 1$ and $n_0 = n_0(R, C) \in \mathbb{N}$ so that, for any $v \in h_{+,0}^{-m\alpha}$ with*

$$\|v\|_{h_+^{-m\alpha}} \leq R,$$

the spectrum $\text{spec}(T_\alpha(v))$ of $T_\alpha(v)$ satisfies the estimates:

(a) *There are precisely $2n_0$ eigenvalues inside the bounded cone*

$$T_{M,n_0} = \{ \lambda \in \mathbb{C} \mid |Im \lambda| - M \leq Re \lambda \leq ((2n_0)^{2m} - (2n_0)^m) \pi^{2m} \}.$$

(b) *For any $n > n_0$ the pairs of eigenvalues $\lambda_{2n-1}(\alpha, m, v)$, $\lambda_{2n}(\alpha, m, v)$ are inside a disc around $(2n-1)^{2m} \pi^{2m}$:*

$$\begin{aligned} |\lambda_{2n-1}(\alpha, m, v) - (2n-1)^{2m} \pi^{2m}| &< 3^m \sqrt{2}CR(2n-1)^{m\alpha}, \\ |\lambda_{2n}(\alpha, m, v) - (2n-1)^{2m} \pi^{2m}| &< 3^m \sqrt{2}CR(2n-1)^{m\alpha}. \end{aligned}$$

So, by Theorem 3.5, it follows that uniformly for bounded sets of sequence v in $h_+^{-m\alpha}$ the one-term asymptotic formulae

$$(3.5) \quad \lambda_{2n-1}(\alpha, m, v), \lambda_{2n}(\alpha, m, v) = (2n-1)^{2m} \pi^{2m} + O(n^{m\alpha}).$$

are hold.

In the next Sections we improve the asymptotic formulae (3.5). For this purpose we will consider vertical strips $\text{Vert}_n^m(r_n)$ with a circle of radius $r_n = (2n-1)^m$ around $(2n-1)^{2m} \pi^{2m}$ removed, and the following estimate for the operators S_λ will be useful:

Lemma 3.6. *Let $m \in \mathbb{N}$, $\alpha \in [0, 1)$, and $\varepsilon > 0$. Then there exists $C = C(\alpha, m, \varepsilon)$ such that, for any $v \in h_{+,0}^{-m\alpha}$*

$$\left\| \left(\sup_{\lambda \in \text{Vert}_n^m((2n-1)^m)} \|S_\lambda\|_{\mathcal{L}(h_-^0)} \right)_{n \geq 1} \right\|_{h^{m(1-\alpha-\varepsilon)}} \leq C \|v\|_{h_+^{-m\alpha}}.$$

Proof. For any given $n \geq 1$, split the operator S_λ ,

$$S_\lambda = \sum_{j=1}^6 I_{A_n^{(j)}} S_\lambda,$$

where

$$I_A : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{R}$$

denotes the characteristic function of a set

$$A \subseteq \mathbb{Z} \times \mathbb{Z}$$

and $A^{(j)} \equiv A_n^{(j)}$ is the following decomposition of $\mathbb{Z} \times \mathbb{Z}$

$$\begin{aligned} A^{(1)} &:= \{(k, j) \in \mathbb{Z}^2 \mid 2k-1, 2j-1 \in \{\pm(2n-1)\}\}; \\ A^{(2)} &:= \{(k, j) \in \mathbb{Z}^2 \mid k, j \in \mathbb{Z} \setminus \{\pm(2n-1)\}\}; \\ A^{(3)} &:= \{(k, j) \in \mathbb{Z}^2 \mid k = n, j \neq \pm(2n-1)\}; \\ A^{(4)} &:= \{(k, j) \in \mathbb{Z}^2 \mid 2k-1 = -(2n-1), j \neq \pm(2n-1)\}; \\ A^{(5)} &:= \{(k, j) \in \mathbb{Z}^2 \mid 2k-1 \neq \pm(2n-1), 2j-1 = 2n-1\}; \\ A^{(6)} &:= \{(k, j) \in \mathbb{Z}^2 \mid 2k-1 \neq \pm(2n-1), 2j-1 = -(2n-1)\}. \end{aligned}$$

Then

$$\sup_{\lambda \in \text{Vert}_n^m((2n-1)^m)} \|S_\lambda\|_{\mathcal{L}(h_-^0)} \leq \sum_{j=1}^6 \sup_{\lambda \in \text{Vert}_n^m((2n-1)^m)} \|I_{A_n^{(j)}} S_\lambda\|_{\mathcal{L}(h_-^0)}$$

and each term in the latter sum is treated separately.

As $v(0) = 0$, we have, for any $\lambda \in \text{Vert}_n^m((2n-1)^m)$

$$\begin{aligned} \|I_{A_n^{(1)}} S_\lambda\|_{\mathcal{L}(h_-^0)} &\leq \left(\sum_{(2k-1, 2j-1) = \pm(2n-1, -2n+1)} \frac{|v(2k-2j)|^2}{|\lambda - (2k-1)^{2m} \pi^{2m}| |\lambda - (2j-1)^{2m} \pi^{2m}|} \right)^{1/2} \\ &\leq \frac{1}{(2n-1)^m} (|v(2(2n-1))|^2 + |v(-2(2n-1))|^2)^{1/2} \in h^{m(1-\alpha)}. \end{aligned}$$

The operators $I_{A_n^{(j)}} S_\lambda$ for $3 \leq j \leq 6$ are estimated similar. Let us consider e.g. the case $j = 3$. For non-negative sequences a in h^0 with $a(n) = 0$, $\forall n \leq 0$ and b, c in h_-^0 , we have

$$\begin{aligned} &\sum_n a(n) \langle n \rangle^{m(1-\alpha)} \sup_{\lambda \in \text{Vert}_n^m((2n-1)^m)} \left| \sum_k \left(I_{A_n^{(3)}} S_{m\lambda} b \right) (2k-1) \cdot c(2k-1) \right| \\ &\leq \sum_{n \geq 1} \sum_{2k-1=2n-1, 2j-1 \neq \pm(2n-1)} a(2n-1) \langle 2n-1 \rangle^{m(1-\alpha)} \\ &\quad \cdot \sup_{\lambda \in \text{Vert}_n^m((2n-1)^m)} \frac{|v(2k-2j)|}{|\lambda - (2k-1)^{2m} \pi^{2m}|^{1/2} |\lambda - (2j-1)^{2m} \pi^{2m}|^{1/2}} b(2j-1) c(2k-1) \\ &\leq \frac{\sqrt{3}}{\pi^m} \sup_{n \geq 1, j \neq \pm(2n-1)} \frac{\langle 2n-1 \rangle^{m(1-\alpha)} \langle 2n-2j \rangle^{m\alpha}}{(2n-1)^{m/2} |(2j-1)^{2m} - (2n-1)^{2m}|^{1/2}} \\ &\quad \cdot \sum_{n \geq 1, 2j-1 \neq \pm(2n-1)} \frac{|v(2n-2j)|}{\langle 2n-2j \rangle^{m\alpha}} a(2n-1) b(2j-1) c(2n-1), \end{aligned}$$

where for the last inequality we use (3.4).

By the Cauchy-Schwartz inequality and the following estimate obtained from Lemma 3.3(b)

$$\sup_{n \geq 1, j \neq \pm n} \frac{\langle n \rangle^{m(1-\alpha)} \langle n-j \rangle^{m\alpha}}{n^{m/2} |j^{2m} - n^{2m}|^{1/2}} \leq 4^m n^{(-1/2+1/2m)}$$

one gets

$$\begin{aligned} \sum_n a(n) \langle n \rangle^{m(1-\alpha)} \sup_{\lambda \in \text{Vert}_n^m((2n-1)^m)} \left| \sum_k \left(I_{A_n^{(3)}} S_\lambda b \right) (2k-1) \cdot c(2k-1) \right| \\ \leq \frac{4^m \sqrt{3}}{\pi^m} \|v\|_{h_+^{-m\alpha}} \|a\|_{h_-^0} \|b\|_{h_-^0} \|c\|_{h_-^0}. \end{aligned}$$

It remains to estimate $\|I_{A_n^{(2)}} S_\lambda\|_{\mathcal{L}(h_-^0)}$. For non-negative sequences a in h^0 with $a(n) = 0$, $\forall n \leq 0$ and b, c in h_-^0 , and any $\varepsilon > 0$ (without loss of generality we assume $1 - \alpha - \varepsilon \geq 0$), one obtains in the same fashion as above

$$\begin{aligned} \sum_n a(n) \langle n \rangle^{m(1-\alpha-\varepsilon)} \sup_{\lambda \in \text{Vert}_n^m((2n-1)^m)} \left| \sum_k \left(I_{A_n^{(2)}} S_{m\lambda} b \right) (2k-1) \cdot c(2k-1) \right| \\ \leq \frac{3}{\pi^{2m}} \sum_{n \geq 1, 2k-1, 2j-1 \neq \pm(2n-1)} R_m(n, k, j) \frac{|v(2k-2j)|}{\langle 2k-2j \rangle^{m\alpha}} a(n) b(2j-1) c(2k-1), \end{aligned}$$

where

$$R_m(n, k, j) := \frac{\langle n \rangle^{m(1-\alpha-\varepsilon)} \langle 2k-2j \rangle^{m\alpha}}{|(2k-1)^{2m} - n^{2m}|^{1/2} |(2j-1)^{2m} - n^{2m}|^{1/2}}.$$

The latter sum is estimated using Cauchy-Schwartz inequality. To estimate $R_m(n, k, j)$, we split up $A_n^{(2)} = \{(2k-1, 2j-1) \in \mathbb{Z}^2 \mid 2k-1, 2j-1 \neq \pm(2n-1)\}$. First notice that, as $R_m(n, k, j)$ is symmetric in k and j , it suffices to consider the case $|2j-1| \leq |2k-1|$. Then, using

$$\langle k-j \rangle^{m\alpha} \leq 2^{m\alpha} \langle k \rangle^{m\alpha}$$

and

$$\langle n \rangle^{m(1-\alpha-\varepsilon)} \leq 2^{m(1-\alpha-\varepsilon)} n^{m(1-\alpha-\varepsilon)},$$

we have got

$$R_m(n, k, j) \leq 2^m \frac{n^{m(1-\alpha-\varepsilon)} \langle 2k-1 \rangle^{m\alpha}}{|(2k-1)^{2m} - n^{2m}|^{1/2} |(2j-1)^{2m} - n^{2m}|^{1/2}}.$$

For the subsets $A_n^{(\pm, \pm)} \cap \{|2j-1| \leq |2k-1| \leq 2n\}$ of $A_n^{(2)}$,

$$A_n^{(\pm, \pm)} := \{(2k-1, 2j-1) \in A_n^{(2)} \mid \pm(2k-1) \geq 0; \pm(2j-1) \geq 0\},$$

we argue similarly. Consider e.g. $A_n^{(-, +)}$. Then $|2k-1| + n = |2k-1-n|$ and $|2j-1| + n = |2j-1+n|$, hence

$$n^{m(1-\alpha-\varepsilon)} \langle 2k-1 \rangle^{m\alpha} \leq n^{m(1-\alpha-\varepsilon)} (1+2n)^{m\alpha} \leq 3^{m\alpha} n^{m(1-\varepsilon)} :$$

1) $m = 2l + 1$, $l \in \mathbb{N}$

$$\begin{aligned} n^{m(1-\alpha-\varepsilon)} \langle 2k-1 \rangle^{m\alpha} &\leq 3^{m\alpha} n^{m(1-\varepsilon)} \leq 3^{m\alpha} |(2k-1)^m - n^m|^{(1-\varepsilon)/2} |(2j-1)^m + n^m|^{(1-\varepsilon)/2} \\ &\leq 3^{m\alpha} |(2k-1)^m - n^m|^{1/2} |(2k-1)^m + n^m|^{-\varepsilon/2} \\ &\quad \cdot |(2j-1)^m + n^m|^{1/2} |(2j-1)^m - n^m|^{-\varepsilon/2}, \end{aligned}$$

which leads to

$$R_m(n, k, j) \leq 6^m |(2k-1)^m + n^m|^{-(1+\varepsilon)/2} |(2j-1)^m - n^m|^{-(1+\varepsilon)/2};$$

2) $m = 2l$, $l \in \mathbb{N}$

$$\begin{aligned} n^{m(1-\alpha-\varepsilon)} \langle 2k-1 \rangle^{m\alpha} &\leq 3^{m\alpha} n^{m(1-\varepsilon)} \leq 3^{m\alpha} |(2k-1)^m + n^m|^{(1-\varepsilon)/2} |(2j-1)^m + n^m|^{(1-\varepsilon)/2} \\ &\leq 3^{m\alpha} |(2k-1)^m + n^m|^{1/2} |(2k-1)^m - n^m|^{-\varepsilon/2} \\ &\quad |(2j-1)^m + n^m|^{1/2} |(2j-1)^m - n^m|^{-\varepsilon/2}, \end{aligned}$$

which leads to

$$R_m(n, k, j) \leq 6^m |(2k-1)^m - n^m|^{-(1+\varepsilon)/2} |(2j-1)^m - n^m|^{-(1+\varepsilon)/2}.$$

Therefore

$$R_m(n, k, j) \leq 6^m |(-1)^{m+1} (2k-1)^m + n^m|^{-(1+\varepsilon)/2} |(2j-1)^m - n^m|^{-(1+\varepsilon)/2}.$$

By the Cauchy-Schwartz inequality one then gets

$$\begin{aligned}
& \sum_{n \geq 1} \sum_{2k-1, 2j-1 \neq \pm(2n-1)} |(-1)^{m+1}(2k-1)^m - n^m|^{-(1+\varepsilon)} |(2j-1)^m - n^m|^{-(1+\varepsilon)} \\
& \cdot \frac{|v(2k-2j)|}{\langle 2k-2j \rangle^{m\alpha}} a(n) b(2j-1) c(2k-1) \\
(3.6) \quad & \leq \left(\sum_{n \geq 1} a^2(n) \sum_{2j-1 \neq \pm(2n-1)} |(2j-1)^m - n^m|^{-(1+\varepsilon)} \sum_{2k-1 \neq \pm(2n-1)} \langle 2k-2j \rangle^{-2m\alpha} |v(2k-2j)|^2 \right)^{1/2} \\
& \cdot \left(\sum_j b^2(2j-1) \sum_k c^2(2k-1) \sum_{n \geq 1, 2k-1 \neq \pm(2n-1)} |(2k-1)^m - n^m|^{-(1+\varepsilon)} \right)^{1/2} \\
& \leq C \|v\|_{h_+^{-m\alpha}} \|a\|_{h^0} \|b\|_{h_-^0} \|c\|_{h_-^0}.
\end{aligned}$$

Next consider the subsets $A_n^{(\pm, \pm)} \cap \{|2j-1| \leq |2k-1|; |2k-1| > 2n\}$ of $A_n^{(2)}$. Again we argue similarly for each of these subsets. Consider e.g. $A_n^{(+, +)}$. In the case $\alpha \in [0, 1/2)$, choose without loss of generality $\varepsilon > 0$ with $\frac{1}{2} - \alpha - \frac{\varepsilon}{2} \geq 0$. Then

$$\begin{aligned}
n^{m(1-\alpha-\varepsilon)} \langle 2k-1 \rangle^{m\alpha} & \leq 2^{m\alpha} n^{m(1-\varepsilon)/2} n^{m(1-2\alpha-\varepsilon)/2} |2k-1|^{m\alpha} \\
& \leq 2^{m\alpha} |(2j-1)^m + n^m|^{(1-\varepsilon)/2} |(2k-1)^m + n^m|^{(1-2\alpha-\varepsilon)/2} |(2k-1)^m + n^m|^\alpha \\
& \leq 2^{m\alpha} |(2k-1)^m - n^m|^{1/2} |(2k-1)^m + n^m|^{-\varepsilon/2} \\
& \cdot |(2j-1)^m + n^m|^{1/2} |(2j-1)^m - n^m|^{-\varepsilon/2}
\end{aligned}$$

and we get

$$R_m(n, k, j) \leq 4^m |(2k-1)^m - n^m|^{-(1+\varepsilon)/2} |(2j-1)^m - n^m|^{-(1+\varepsilon)/2},$$

and thus obtain estimate of the type (3.6).

In the case $\alpha \in [1/2, 1)$, since $n \leq |2k-1+n|$,

$$n^{m(1-\alpha-\varepsilon)} \langle 2k-1 \rangle^{m\alpha} \leq 2^{m\alpha} |(2k+1)^m + n^m|^{(1-\alpha-\varepsilon)} |2k-1|^{m\alpha}$$

so using that $|k|^{m(\alpha-1/2)} \leq |(2k-1)^m + n^m|^{(\alpha-1/2)}$ and $|k|^{m/2} \leq 3^{m/2} |(2k-1)^m - n^m|$, we get

$$\begin{aligned}
n^{m(1-\alpha-\varepsilon)} \langle 2k-1 \rangle^{m\alpha} & \leq 2^{m\alpha} |(2k-1)^m + n^m|^{(1-\alpha-\varepsilon)} |2k-1|^{m\alpha} \\
& \leq 2^{m\alpha} |(2k-1)^m + n^m|^{(1-\alpha-\varepsilon)} 3^{m/2} |(2k-1)^m - n^m|^{1/2} |(2k-1)^m + n^m|^{(\alpha-1/2)} \\
& \leq (2\sqrt{3})^m |k|^{2m} - n^{2m}|^{1/2} |k^m + n^m|^{-\varepsilon} \\
& \leq (2\sqrt{3})^m |(2k-1)^{2m} - n^{2m}|^{1/2} |(2j-1)^m - n^m|^{-\varepsilon/2} |(2j-1)^m + n^m|^{-\varepsilon/2},
\end{aligned}$$

where we use that $|(2j-1)^m \pm n^m| \leq |(2k-1)^m + n^m|$. This yields

$$R_m(n, k, j) \leq 8^m |(2j-1)^m + n^m|^{-(1+\varepsilon)/2} |(2j-1)^m - n^m|^{-(1+\varepsilon)/2}$$

and therefore we again obtain an estimate of the type (3.6). \square

For later reference, let us denote for given $m \in \mathbb{N}$, $\alpha \in [0, 1)$ and $R > 0$, by $n_* = n_*(\alpha, m, R) \geq 1$ a number with the property that, for any $v \in h_{+,0}^{-m\alpha}$ with

$$\|v\|_{h_+^{-m\alpha}} \leq R$$

we have got

$$(3.7) \quad \sup_{\lambda \in \text{Vert}_n^m((2n-1)^m)} \|S_\lambda\|_{\mathcal{L}(h_-^0)} \leq \frac{1}{2}, \quad \forall n \geq n_*.$$

4. ASYMPTOTIC ESTIMATES OF τ_n

In this and the next Sections we establish asymptotic estimates for the eigenvalues λ_n . They are obtained by separately considering the mean τ_n and the difference γ_n of a pair λ_{2n} and λ_{2n-1} ,

$$\tau_n := \frac{\lambda_{2n} + \lambda_{2n-1}}{2}, \quad \gamma_n := \lambda_{2n} - \lambda_{2n-1}.$$

In this Section we establish

Theorem 4.1. *Let $m \in \mathbb{N}$, $\alpha \in [0, 1)$, and $\varepsilon > 0$. Then, uniformly for bounded sets of v in $h_{+,0}^{-m\alpha}$,*

$$\tau_n = (2n-1)^{2m} \pi^{2m} + h^{m(1-2\alpha-\varepsilon)}(n).$$

The assertion of Theorem 4.1 is a consequence of Lemma 4.2 and Lemma 4.3 below. Let $R > 0$ and $v \in h_{+,0}^{-m\alpha}$ with

$$\|v\|_{h_{+,0}^{-m\alpha}} \leq R.$$

For $n \geq n_* = n_*(\alpha, m, R)$ with n_* chosen as in (3.7), define the Riesz projectors

$$P_n := \frac{1}{2\pi i} \int_{\Gamma_n} (\lambda - A^m - B(v))^{-1} d\lambda \in \mathcal{L}(h_-^{-m\alpha}),$$

$$P_n^0 := \frac{1}{2\pi i} \int_{\Gamma_n} (\lambda - A^m)^{-1} d\lambda \in \mathcal{L}(h_-^{-m\alpha}),$$

where Γ_n is the positively oriented contour given by

$$\Gamma_n = \{\lambda \in \mathbb{C} \mid |\lambda - (2n-1)^{2m} \pi^{2m}| = (2n-1)^m\}.$$

The corresponding Riesz spaces are the ranges of these projectors,

$$E_n := P_n(h_-^{-m\alpha}), \quad \text{and} \quad E_n^0 := P_n^0(h_-^{-m\alpha}).$$

Both E_n and E_n^0 are two-dimensional subspaces of h_-^0 , and P_n as well as $(A^m + B(v))P_n$ can be considered as operators from $\mathcal{L}(h_-^0)$. Their traces can be computed to be

$$Tr(P_n) = 2, \quad Tr((A^m + B(v))P_n) = 2\tau_n.$$

Similarly, we have

$$Tr(P_n^0) = 2, \quad Tr(A^m P_n^0) = 2(2n-1)^{2m} \pi^{2m}$$

and thus obtain

$$2\tau_n - 2(2n-1)^{2m} \pi^{2m} = Tr((A^m + B(v))P_n) - Tr(A^m P_n^0) = Tr(Q_n),$$

where Q_n is the operator

$$Q_n := (A^m + B(v) - (2n-1)^{2m} \pi^{2m})P_n - (A^m - (2n-1)^{2m} \pi^{2m})P_n^0 \in \mathcal{L}(h_-^0).$$

Substituting the formula for P_n and P_n^0 one gets

$$Q_n = \frac{1}{2\pi i} \int_{\Gamma_n} (\lambda - (2n-1)^{2m} \pi^{2m}) ((\lambda - A^m - B(v))^{-1} - (\lambda - A^m)^{-1}) d\lambda$$

$$= \frac{1}{2\pi i} \int_{\Gamma_n} (\lambda - (2n-1)^{2m} \pi^{2m}) (\lambda - A^m - B(v))^{-1} B(v) (\lambda - A^m)^{-1} d\lambda \in \mathcal{L}(h_-^0).$$

Remark that

$$Q_n^0(2k, 2l) = 0, \quad k, l \in \mathbb{Z}.$$

Write $Q_n = Q_n^0 + Q_n^1$ with

$$Q_n^0 := \frac{1}{2\pi i} \int_{\Gamma_n} (\lambda - (2n-1)^{2m} \pi^{2m}) (\lambda - A^m)^{-1} B(v) (\lambda - A^m)^{-1} d\lambda \in \mathcal{L}(h_-^0),$$

which leads to the following expression for τ_n ,

$$(4.1) \quad \tau_n = (2n-1)^{2m} \pi^{2m} + \frac{1}{2} Tr(Q_n^0) + \frac{1}{2} Tr(Q_n^1).$$

To compute

$$Tr(Q_n^0) = \sum_{k \in \mathbb{Z}} Q_n^0(k, k) = \sum_{k \in \mathbb{Z}} Q_n^0(2k-1, 2k-1)$$

we need the following

Lemma 4.2. *Let $m \in \mathbb{N}$, and $\alpha \in [0, 1)$. For any $v \in h_{+,0}^{-m\alpha}$ with $n \geq 1$ and $k, l \in \mathbb{Z}$,*

$$Q_n^0(2k-1, 2l-1) = \begin{cases} v(\pm 2(2n-1)) & \text{if } (k, l) = (n, -n+1) \text{ or } (k, l) = (-n+1, n); \\ 0 & \text{otherwise.} \end{cases}$$

Proof. For $k, l \in \mathbb{Z}$ and $n \geq 1$, we have

$$\begin{aligned} Q_n^0(2k-1, 2l-1) &= \frac{1}{2\pi i} \int_{\Gamma_n} \frac{\lambda - (2n-1)^{2m} \pi^{2m}}{(\lambda - (2k-1)^{2m} \pi^{2m})(\lambda - (2l-1)^{2m} \pi^{2m})} v(2k-2l) d\lambda \\ &= v(2k-2l) \frac{1}{2\pi i} \int_{\Gamma_n} \frac{\lambda - (2n-1)^{2m} \pi^{2m}}{(\lambda - (2k-1)^{2m} \pi^{2m})(\lambda - (2l-1)^{2m} \pi^{2m})} d\lambda \end{aligned}$$

and the claimed statement follows from

$$\frac{1}{2\pi i} \int_{\Gamma_n} \frac{\lambda - (2n-1)^{2m} \pi^{2m}}{(\lambda - (2k-1)^{2m} \pi^{2m})(\lambda - (2l-1)^{2m} \pi^{2m})} d\lambda = \begin{cases} 1 & \text{if } (k, l) \in \{n, -n+1\}, \\ 0 & \text{otherwise.} \end{cases}$$

□

Lemma 4.2 implies that

$$\text{Tr}(Q_n^0) = 0,$$

and, moreover, $\text{range}(Q_n^0) \subseteq E_n^0$. So, since $\text{range}(Q_n) \subseteq \text{span}(E_n \cup E_n^0)$ by definition, we conclude that

$$\text{range}(Q_n^1) \subseteq \text{span}(E_n \cup E_n^0)$$

as well. Hence $\text{range}(Q_n^1)$ is at most dimension four and

$$|\text{Tr}(Q_n^1)| \leq 4 \|Q_n^1\|_{\mathcal{L}(h_-^0)}.$$

Theorem 4.1 then follows from (4.1) together with

Lemma 4.3. *Let $m \in \mathbb{N}$, $\alpha \in [0, 1)$, $R > 0$ and $\varepsilon > 0$. Then there exists $C = C(\alpha, m, \varepsilon)$ so that, for any $v \in h_{+,0}^{-m\alpha}$ with*

$$\begin{aligned} \|v\|_{h_+^{-m\alpha}} &\leq R, \\ \left\| \left(\|Q_n^1\|_{\mathcal{L}(h_-^0)} \right)_{n \geq n_*} \right\|_{h^{m(1-2\alpha-\varepsilon)}} &\leq C \|v\|_{h_+^{-m\alpha}}, \end{aligned}$$

where $n_* = n_*(\alpha, m, R)$ is given by (3.7).

Proof. By (3.2), for any $\lambda \in \Gamma_n$, $(\lambda - A^m - B(v))^{-1}$ is given by $A_\lambda^{-m/2} (I_\lambda - S_\lambda)^{-1} A_\lambda^{-m/2}$. Hence

$$\begin{aligned} Q_n^1 &= \frac{1}{2\pi i} \int_{\Gamma_n} (\lambda - (2n-1)^{2m} \pi^{2m}) ((\lambda - A^m - B(v))^{-1} - (\lambda - A^m)^{-1} B(v) (\lambda - A^m)^{-1}) d\lambda \\ &= \frac{1}{2\pi i} \int_{\Gamma_n} (\lambda - (2n-1)^{2m} \pi^{2m}) A_\lambda^{-m/2} (I_\lambda - S_\lambda)^{-1} S_\lambda I_\lambda^{-1} S_\lambda I_\lambda^{-1} A_\lambda^{-m/2} d\lambda. \end{aligned}$$

Using (3.1) and (3.4) one shows that, for $\lambda \in \text{Vert}_n^m(r_n)$ ($n \geq \frac{8m^2+4m-7}{2(8m-7)}$, $0 < r_n < (2n-1)^m \pi^{2m}$),

$$(4.2) \quad \|A_\lambda^{-m/2}\|_{\mathcal{L}(h_-^0)} \leq r_n^{-1/2} + \frac{\sqrt{3}}{\pi^m} (2n-1)^{-m+1/2}.$$

Together with Lemma 3.6 the claimed statement then follows. □

5. ASYMPTOTIC ESTIMATES OF γ_n

To state the asymptotic of γ_n let introduce, for $v \in h_{+,0}^{-m\alpha}$, the sequence

$$w := \frac{1}{\pi^{2m}} \frac{v}{k^m} * \frac{v}{k^m} \in h_+^{mt} \quad \text{with} \quad w(2n) = \frac{1}{\pi^{2m}} \sum_{k \neq \pm n} \frac{v(n-k)}{(n-k)^m} \cdot \frac{v(n+k)}{(n+k)^m}$$

Note that $\frac{v}{k^m} \in h_+^{m(1-\alpha)}$. By the Convolution Lemma, this implies that $w \in h_+^{mt}$, where for $\alpha \in [0, 1 - 1/2m)$ we have got $t = (1 - \alpha)$, and, for $\alpha \in [1 - 1/2m, 1)$, any $t < 2(1 - \alpha) - 1/2m$ can be chosen. In particular, we can always chose $t > -\alpha$.

Further, let consider the sequence $(l(n))_{n \in \mathbb{Z}}$, such that

$$l(2n) := \frac{1}{\pi^{2m}} \sum_{k \neq \pm n} \frac{v(n-k)}{n^m - k^m} \cdot \frac{v(n+k)}{n^m + k^m}.$$

We have got $(l(n))_{n \in \mathbb{Z}} \in h_+^{mt}$, where t is chosen as above, and

$$(5.1) \quad \|l\|_{h_+^{mt}} \leq \text{Const} \|w\|_{h_+^{mt}}.$$

Theorem 5.1. *Let $m \in \mathbb{N}$, $\alpha \in [0, 1)$ and $\varepsilon > 0$. Then, uniformly on bounded sets of v in $h_{+,0}^{-m\alpha}$,*

$$\left(\min_{\pm} \left| \gamma_n \pm 2\sqrt{(v+l)(-2(2n-1)) \cdot (v+l)(2(2n-1))} \right| \right)_{n \geq 1} \in h^{m(1-2\alpha-\varepsilon)}.$$

Remark. *An asymptotic estimate only involving v but not l is of the form*

$$\left(\min_{\pm} \left| \gamma_n \pm 2\sqrt{v(-2(2n-1)) \cdot v(2(2n-1))} \right| \right)_{n \geq 1} \in \begin{cases} h^{m(1/2-\alpha)} & \text{if } \alpha \in [0, 1/2), \\ h^{m(1-2\alpha-\varepsilon)} & \text{if } \alpha \in [1/2, 1). \end{cases}$$

Proof. To prove Theorem 5.1, consider for $n_* = n_*(\alpha, m, R)$ and $v \in h_{+,0}^{-m\alpha}$ with

$$\|v\|_{h_+^{-m\alpha}} \leq R,$$

the restriction K_n of $A^m + B(v) - \tau_n$ to the Riesz space E_n ,

$$K_n : E_n \longrightarrow E_n.$$

The eigenvalues of K_n are $\pm \frac{\gamma_n}{2}$, hence

$$\det(K_n) = -\left(\frac{\gamma_n}{2}\right)^2.$$

We need the following auxiliary result:

Lemma 5.2. *Let $m \in \mathbb{N}$, $\alpha \in [0, 1)$, $R > 0$ and $\varepsilon > 0$. Then there exists $C > 0$ so that, for any $v \in h_{+,0}^{-m\alpha}$ with*

$$\|v\|_{h_+^{-m\alpha}} \leq R$$

we have

- (i) $\|P_n\|_{\mathcal{L}(h_-^0)} \leq C, \quad \forall n \geq n_*;$
- (ii) $\left(\|P_n - P_n^0\|_{\mathcal{L}(h_-^0)} \right)_{n \geq n_*} \in h^{m(1-\alpha-\varepsilon)}.$

Proof. Recall that, for $n \geq n_*$

$$P_n = \frac{1}{2\pi i} \int_{\Gamma_n} A_\lambda^{-m/2} (I_\lambda - S_\lambda)^{-1} A_\lambda^{-m/2} d\lambda$$

and

$$P_n - P_n^0 = \frac{1}{2\pi i} \int_{\Gamma_n} A_\lambda^{-m/2} (I_\lambda - S_\lambda)^{-1} S_\lambda I_\lambda^{-1} A_\lambda^{-m/2} d\lambda.$$

The claimed estimates then follow from (3.1) and Lemma 3.6. \square

Choose, if necessary, n_* larger so that

$$(5.2) \quad \|P_n - P_n^0\|_{\mathcal{L}(h_-^0)} \leq \frac{1}{2}, \quad \forall n \geq n_*.$$

One verifies easily that

$$Q_n := (P_n - P_n^0)^2$$

commutes with P_n and P_n^0 . Hence Q_n leaves both Riesz spaces E_n and E_n^0 invariant. The operator Q_n is used to define, for $n \geq n_*$, the restriction of the transformation operator

$$(Id - Q_n)^{-1/2} (P_n P_n^0 + (Id - P_n)(Id - P_n^0))$$

to E_n^0 (cf.[3]),

$$U_n := (Id - Q_n)^{-1/2} P_n P_n^0 : E_n^0 \longrightarrow E_n,$$

where $(Id - Q_n)^{-1/2}$ is given by the binomial formula

$$(5.3) \quad (Id - Q_n)^{-1/2} = \sum_{l \geq 0} \binom{-1/2}{l} (-Q_n)^l.$$

One verifies that U_n is invertible with the inverse given by

$$(5.4) \quad U_n^{-1} := P_n^0 P_n (Id - Q_n)^{-1/2}.$$

As a consequence,

$$\det(U_n^{-1} K_n U_n) = -\left(\frac{\gamma_n}{2}\right)^2.$$

To estimate $\det(U_n^{-1} K_n U_n)$, write

$$U_n^{-1} K_n U_n = P_n^0 P_n K_n P_n P_n^0 + R_n^{(1)} + R_n^{(2)},$$

where

$$R_n^{(1)} := (U_n^{-1} - P_n P_n^0) K_n P_n P_n^0; \quad R_n^{(2)} := U_n^{-1} K_n (U_n - P_n P_n^0).$$

The term

$$P_n^0 P_n K_n P_n P_n^0 = P_n^0 P_n (A^m + B(v) - \tau_n) P_n^0$$

is split up further,

$$\begin{aligned} P_n^0 P_n (A^m + B(v) - \tau_n) P_n^0 &= P_n^0 (A^m + B(v) - \tau_n) P_n^0 + P_n^0 (P_n - P_n^0) (A^m + B(v) - \tau_n) P_n^0 \\ &= P_n^0 B(v) P_n^0 + L_n^{(1)} + P_n^0 (P_n - P_n^0) B(v) P_n^0 + R_n^{(3)}, \end{aligned}$$

where $L_n^{(1)}$ is a diagonal operator (use $A^m P_n = n^{2m} \pi^{2m} P_n^0$)

$$L_n^{(1)} := P_n^0 ((2n-1)^{2m} \pi^{2m} - \tau_n) P_n^0$$

and

$$R_n^{(3)} := P_n^0 (P_n - P_n^0) ((2n-1)^{2m} \pi^{2m} - \tau_n) P_n^0.$$

As a 2×2 matrix, $B_n := P_n^0 B(v) P_n^0$ is given by

$$\begin{pmatrix} B_n(2n-1, 2n-1) & B_n(2n-1, -2n+1) \\ B_n(-2n+1, 2n-1) & B_n(-2n+1, -2n+1) \end{pmatrix} = \begin{pmatrix} 0 & v(2(2n-1)) \\ v(-2(2n-1)) & 0 \end{pmatrix}.$$

To obtain a satisfactory estimate for

$$\det(U_n^{-1} K_n U_n),$$

we have to substitute an expansion of $(P_n - P_n^0)$ into $P_n^0 (P_n - P_n^0) B(v) P_n^0$ and split the main term into a diagonal part $L_n^{(2)}$ and an off-diagonal part. Let us explain this in more detail. Write

$$\begin{aligned} P_n - P_n^0 &= \frac{1}{2\pi i} \int_{\Gamma_n} (\lambda - A^m)^{-1} B(v) (\lambda - A^m)^{-1} d\lambda \\ &\quad + \frac{1}{2\pi i} \int_{\Gamma_n} (\lambda - A^m)^{-1} B(v) (\lambda - A^m)^{-1} B(v) (\lambda - A^m - B(v))^{-1} d\lambda, \end{aligned}$$

which leads to

$$P_n^0 (P_n - P_n^0) B(v) P_n^0 = \mathcal{S}_n + R_n^{(4)},$$

where

$$R_n^{(4)} := \frac{1}{2\pi i} \int_{\Gamma_n} P_n^0 A_\lambda^{-m/2} I_\lambda^{-1} S_\lambda I_\lambda^{-1} S_\lambda (I_\lambda - S_\lambda)^{-1} S_\lambda A_\lambda^{m/2} P_n^0 d\lambda$$

and

$$(5.5) \quad \mathcal{S}_n := \frac{1}{2\pi i} \int_{\Gamma_n} P_n^0 (\lambda - A^m)^{-1} B(v) (\lambda - A^m)^{-1} B(v) P_n^0 d\lambda.$$

As a 2×2 matrix,

$$\mathcal{S}_n = \begin{pmatrix} \mathcal{S}_n(2n-1, 2n-1) & \mathcal{S}_n(2n-1, -2n+1) \\ \mathcal{S}_n(-2n+1, 2n-1) & \mathcal{S}_n(-2n+1, -2n+1) \end{pmatrix}$$

is of the form

$$\mathcal{S}_n = K_n^{(2)} + \begin{pmatrix} 0 & l(2(2n-1)) \\ l(-2(2n-1)) & 0 \end{pmatrix},$$

where $K_n^{(2)}$ is the diagonal part of \mathcal{S}_n .

Combining the computation above, one obtains the following identity

$$(5.6) \quad U_n^{-1} K_n U_n = \begin{pmatrix} 0 & (v+l)(2(2n-1)) \\ (v+l)(-2(2n-1)) & 0 \end{pmatrix} + L_n + R_n,$$

where L_n is the diagonal matrix

$$L_n = L_n^{(1)} + L_n^{(2)}$$

and R_n is the sum

$$R_n = \sum_{j=1}^4 R_n^{(j)}.$$

The identity (5.6) leads to the following expression for the determinant

$$-\left(\frac{\gamma_n}{2}\right)^2 = \det(U_n^{-1} K_n U_n) = -(v+l)(2(2n-1))(v+l)(-2(2n-1)) - r_n,$$

where the error r_n is given by

$$\begin{aligned} r_n = & -(K_n(2n-1, 2n-1) + R_n(2n-1, 2n-1))(K_n(-2n+1, -2n+1) + R_n(-2n+1, -2n+1)) \\ & + (v+l)(2(2n-1))R_n(-2n+1, 2n-1) + (v+l)(-2(2n-1))R_n(2n-1, -2n+1) \\ & + R_n(-2n+1, 2n-1)R_n(2n-1, -2n+1). \end{aligned}$$

Hence

$$\min_{\pm} \left| \frac{\gamma_n}{2} \pm \sqrt{(v+l)(-2(2n-1)) \cdot (v+l)(2(2n-1))} \right| \leq |r_n|^{1/2}.$$

To estimate r_n , use that an entry of a matrix is bounded by its norm. Hence, for some universal constant $C > 0$ and $n \geq n_*$,

$$(5.7) \quad |r_n| \leq C \left(\|K_n\|_{\mathcal{L}(h_-^0)}^2 + \|R_n\|_{\mathcal{L}(h_-^0)}^2 + \sum_{\pm} |(v+l)(\pm 2(2n-1))| \|R_n\|_{\mathcal{L}(h_-^0)} \right).$$

The terms on the right side of the inequality above are estimated separately. By Theorem 4.1,

$$\|K_n^{(1)}\|_{\mathcal{L}(h_-^0)} = |(2n-1)^{2m} \pi^{2m} - \tau_n| = h^{m(1-2\alpha-\varepsilon)}(n).$$

As

$$\|K_n^{(2)}\| = \text{diag}(\mathcal{S}_n(2n-1, 2n-1), \mathcal{S}_n(-2n+1, -2n+1))$$

we have

$$\|K_n^{(2)}\|_{\mathcal{L}(h_-^0)} \leq \|\mathcal{S}_n\|_{\mathcal{L}(h_-^0)}$$

and, by the definition (5.5) of \mathcal{S}_n ,

$$\begin{aligned} \|\mathcal{S}_n\|_{\mathcal{L}(h_-^0)} &= \left\| \frac{1}{2\pi i} \int_{\Gamma_n} P_n^0 A_\lambda^{-m/2} I_\lambda^{-1} S_\lambda I_\lambda^{-1} S_\lambda A_\lambda^{m/2} P_n^0 d\lambda \right\| \\ &\leq (2n-1)^m \left(\sup_{\lambda \in \Gamma_n} \|A_\lambda^{-m/2}\|_{\mathcal{L}(h_-^0)} \|S_\lambda\|_{\mathcal{L}(h_-^0)}^2 \|A_\lambda^{m/2} P_n^0\|_{\mathcal{L}(h_-^0)} \right). \end{aligned}$$

By Lemma 3.6 and (5.1) we then conclude

$$\|K_n^{(2)}\|_{\mathcal{L}(h_-^0)} = h^{m(1-2\alpha-\varepsilon)}(n).$$

By the definition of $R_n^{(1)}$,

$$\|R_n^{(1)}\|_{\mathcal{L}(h_-^0)} \leq \|U_n^{-1} - P_n P_n^0\|_{\mathcal{L}(h_-^0)} \|(A^m + B(v) - \tau_n) P_n\|_{\mathcal{L}(h_-^0)} \|P_n^0\|_{\mathcal{L}(h_-^0)}.$$

We have $\|P_n^0\|_{\mathcal{L}(h_-^0)} = 1$ and

$$\|A^m + B(v) - \tau_n\|_{\mathcal{L}(h_-^0)} = \left\| \frac{1}{2\pi i} \int_{\Gamma_n} (\lambda - \tau_n)(\lambda - A^m - B(v))^{-1} d\lambda \right\|_{\mathcal{L}(h_-^0)} \leq C(2n-1)^{m\alpha},$$

where for the last inequality we use Theorem 3.5 to deform the contour Γ_n to a circle Γ'_n of radius $C(2n-1)^{m\alpha}$ around $(2n-1)^{2m} \pi^{2m}$ and the estimate

$$\|(\lambda - A^m - B(v))^{-1}\|_{\mathcal{L}(h_-^0)} = \|A_\lambda^{m/2} (I_\lambda - S_\lambda)^{-1} A_\lambda^{m/2}\|_{\mathcal{L}(h_-^0)} \leq C(2n-1)^{-m\alpha}, \quad \forall \lambda \in \Gamma'_n.$$

By the formula (5.4) for U_n^{-1} , we have, in view of the binomial formula (5.3) and the definition $Q_n = (P_n - P_n^0)^2$,

$$\|U_n^{-1} - P_n P_n^0\|_{\mathcal{L}(h_-^0)} \leq \|P_n\|_{\mathcal{L}(h_-^0)} \|P_n^0\|_{\mathcal{L}(h_-^0)} \sum_{l \geq 1} \left| \binom{-1/2}{l} \right| \|P_n - P_n^0\|_{\mathcal{L}(h_-^0)}^{2l},$$

where for the last inequality we use lemma 5.2(i) and the estimate

$$\|P_n - P_n^0\|_{\mathcal{L}(h_-^0)} \leq \frac{1}{2}, \quad n \geq n_*.$$

Hence, by Lemma 5.2(ii),

$$\|R_n^{(1)}\|_{\mathcal{L}(h_-^0)} \leq C(2n-1)^{-m\alpha} \|P_n - P_n^0\|_{\mathcal{L}(h_-^0)}^2 = (2n-1)^{m\alpha} (h^{m(1-\alpha-\varepsilon)}(n))^2.$$

Similarly one shows

$$\|R_n^{(2)}\|_{\mathcal{L}(h_-^0)} = (2n-1)^{m\alpha} (h^{m(1-\alpha-\varepsilon)}(n))^2.$$

In view of the definition $R_n^{(3)}$,

$$\|R_n^{(3)}\|_{\mathcal{L}(h_-^0)} \leq C \|P_n - P_n^0\|_{\mathcal{L}(h_-^0)} |(2n-1)^{2m} \pi^{2m} - \tau_n| = (2n-1)^{m\alpha} (h^{m(1-\alpha-\varepsilon)}(n))^2,$$

where we use Lemma 5.2 to estimate $\|P_n - P_n^0\|_{\mathcal{L}(h_-^0)}$ and Theorem 4.1 to bound

$$|(2n-1)^{2m} \pi^{2m} - \tau_n|.$$

Finally, by the definition of $R_n^{(4)}$ and Lemma 3.6,

$$\|R_n^{(4)}\|_{\mathcal{L}(h_-^0)} \leq C(2n-1)^m \|S_\lambda\|_{\mathcal{L}(h_-^0)}^3 \leq (2n-1)^m (h^{m(1-\alpha-\varepsilon/2)}(n))^3 \leq (2n-1)^{m\alpha} (h^{m(1-\alpha-\varepsilon)}(n))^2.$$

Combining the obtained estimates one gets

$$(5.8) \quad \|R_n\|_{\mathcal{L}(h_-^0)} = (2n-1)^{m\alpha} (h^{m(1-\alpha-\varepsilon)}(n))^2,$$

$$(5.9) \quad \|K_n\| = h^{m(1-2\alpha-\varepsilon)}(n).$$

Taking to account that

$$\|l\|_{h_+^{mt}} \leq \text{Const} \|w\|_{h_+^{mt}}, \quad t > -m\alpha,$$

as a consequence we have obtained

$$(5.10) \quad |(v+l)(\pm 2(2n-1))| \|R_n\|_{\mathcal{L}(h_-^0)} = (h^{m(1-2\alpha-\varepsilon)}(n))^2$$

and, in view of (5.7),

$$|r_n|^{1/2} = h^{m(1-2\alpha-\varepsilon)}(n).$$

This proves Theorem 5.1. □

6. THE LIMITING CASE $\alpha = 1$

In this Section the spectral problem

$$Tu = \lambda u$$

for the operator

$$T(v) \equiv T_1(v) = A^m + B(v), \quad v \in h_+^{-m}$$

is studied.

At first, in a straightforward way, one can prove the following two auxiliary lemmas.

Lemma 6.1. *For any $s, t \in \mathbb{R}$ with $s - t \leq 2$ and any*

$$\lambda \in \mathbb{C} \setminus \text{spec}(A^m), \quad m \in \mathbb{N}$$

we have

$$(\lambda - A^m)^{-1} \in \mathcal{L}(h_-^{mt}, h_-^{ms})$$

with norm

$$\|(\lambda - A^m)^{-1}\|_{\mathcal{L}(h_-^{mt}, h_-^{ms})} = \sup_{k \in \mathbb{Z}} \frac{\langle 2k-1 \rangle^{m(s-t)}}{|\lambda - (2k-1)^{2m} \pi^{2m}|} < \infty.$$

Lemma 6.2. *Uniformly for $n \in \mathbb{Z} \setminus \{0\}$ and $\lambda \in \text{Vert}_n^m(r_n)$ the following estimates are valid:*

$$\begin{aligned}
(a) \quad & \|(\lambda - A^m)^{-1}\|_{\mathcal{L}(h_-^m)} = \frac{1}{r_n} O(1), & (a') \quad & \|(\lambda - A^m)^{-1}\|_{\mathcal{L}(h_-^m)} = O(n^{-m}), \\
(b) \quad & \|(\lambda - A^m)^{-1}\|_{\mathcal{L}(h_-^{m,n})} = \frac{1}{r_n} O(1), & (b') \quad & \|(\lambda - A^m)^{-1}\|_{\mathcal{L}(h_-^{m,n})} = O(n^{-m}), \\
(c) \quad & \|(\lambda - A^m)^{-1}\|_{\mathcal{L}(h_-^{m,n}, h_-^m)} = \frac{1}{r_n} O(1), & (c') \quad & \|(\lambda - A^m)^{-1}\|_{\mathcal{L}(h_-^{m,n}, h_-^m)} = O(n^{-m}), \\
(d) \quad & \|(\lambda - A^m)^{-1}\|_{\mathcal{L}(h_-^m, h_-^{m,n})} = \frac{(2n-1)^{2m}}{r_n} O(1), & (d') \quad & \|(\lambda - A^m)^{-1}\|_{\mathcal{L}(h_-^m, h_-^{m,n})} = O(n^m), \\
(e) \quad & \|(\lambda - A^m)^{-1}\|_{\mathcal{L}(h_-^{m,n}, h_-^{m,-n})} = \frac{(2n-1)^m}{r_n} O(1); & (e') \quad & \|(\lambda - A^m)^{-1}\|_{\mathcal{L}(h_-^{m,n}, h_-^{m,-n})} = O(1).
\end{aligned}$$

Theorem 6.3. *Let $m \in \mathbb{N}$, and $v \in h_+^{-m}$. There exist $\varepsilon > 0$, $M \geq 1$ and $n_0 \in \mathbb{N}$ so that for any $w \in h_+^{-m}$ with*

$$\|w - v\|_{h_+^{-m}} \leq \varepsilon$$

the spectrum $\text{spec}(T(w))$ of the operator

$$T(w) = A^m + B(w)$$

consists of a sequence $(\lambda_k(m, w))_{k \geq 1}$ such that:

(a) *There are precisely $2n_0$ eigenvalues inside the bounded cone*

$$T_{M, n_0} = \{\lambda \in \mathbb{C} \mid |Im \lambda| - M \leq Re \lambda \leq ((2n_0)^{2m} - (2n_0)^m) \pi^{2m}\}.$$

(b) *For $n > n_0$ the pairs of eigenvalues $\lambda_{2n-1}(m, w)$, $\lambda_{2n}(m, w)$ are inside a disc around $(2n-1)^{2m} \pi^{2m}$:*

$$\begin{aligned}
|\lambda_{2n-1}(m, w) - (2n-1)^{2m} \pi^{2m}| &< (2n-1)^m, \\
|\lambda_{2n}(m, w) - (2n-1)^{2m} \pi^{2m}| &< (2n-1)^m.
\end{aligned}$$

Proof. Let $v \in h_+^{-m}$. Since the set h_+^m is dense in the space h_+^{-m} , we can represent v in the form

$$v = v_0 + v_1, \quad \text{with } v_0 \in h_+^m \quad \text{and} \quad \|v_1\|_{h_+^{-m}} \leq \varepsilon,$$

where $\varepsilon > 0$ will be find bellow. We will show that for some $M \geq 1$ and $n_0 \in \mathbb{N}$, which are both depending on $\|v_0\|_{h_+^m}$, so that for any $w = v + \tilde{w} \in h_+^{-m}$ with $\|\tilde{w}\|_{h_+^{-m}} \leq \varepsilon$, we have got

$$(6.1) \quad \text{Ext}_M \cup \bigcup_{n \geq n_0} \text{Vert}_n^m((2n-1)^m) \subseteq \text{Resol}(T(w)),$$

where $\text{Resol}(T(w))$ denotes the resolvent set of the operator

$$T(w) = A^m + B(v_0) + B(v_1 + \tilde{w}).$$

At first let consider $\lambda \in \text{Ext}_M$ for $M \geq 1$. Using the Convolution Lemma and the Lemma 6.1 one gets

$$\|B(v_0)(\lambda - A^m)^{-1}\|_{\mathcal{L}(h_-^m)} \leq C_m \|v_0\|_{h_+^m} \|(\lambda - A^m)^{-1}\|_{\mathcal{L}(h_-^m)} = \|v_0\|_{h_+^m} \cdot O(M^{-1}).$$

Hence, for $M \geq 1$ large enough and $\lambda \in \text{Ext}_M$,

$$T_\lambda := \lambda - A^m - B(v_0) = (Id - B(v_0)(\lambda - A^m)^{-1})(\lambda - A^m)$$

is invertible in $\mathcal{L}(h_-^m)$ with inverse

$$(6.2) \quad T_\lambda^{-1} = (\lambda - A^m)^{-1} (Id - B(v_0)(\lambda - A^m)^{-1})^{-1}.$$

So, using the Convolution Lemma and the estimate

$$\|(\lambda - A^m)^{-1}\|_{\mathcal{L}(h_-^m, h_-^m)} = O(1),$$

we have obtained

$$\|B(v_1 + \tilde{w})T_\lambda^{-1}\|_{\mathcal{L}(h_-^m)} = O(\|(v_1 + \tilde{w})\|_{h_+^{-m}}) = O(\varepsilon).$$

Therefore, if $\varepsilon > 0$ is small enough, the resolvent of the operator

$$T(w) = A^m + B(v_0) + B(v_1 + \tilde{w})$$

exists in the space $\mathcal{L}(h_-^{-m})$ for $\lambda \in \text{Ext}_M$ and is given by the formula

$$(6.3) \quad (\lambda - A^m - B(v_0) - B(v_1 + \tilde{w}))^{-1} = (T_\lambda - B(v_1 + \tilde{w}))^{-1} = T_\lambda^{-1} \sum_{k \geq 0} (B(v_1 + \tilde{w})T_\lambda^{-1})^k.$$

Consequently, for M large enough,

$$\text{Ext}_M \subseteq \text{Resol}(T(w)).$$

To treat $\lambda \in \text{Vert}_n^m((2n-1)^m)$, first note that, unfortunately,

$$\|(\lambda - A^m)^{-1}\|_{\mathcal{L}(h_-^{-m}, h_-^m)} = O(n^m),$$

and so we can not argue as above. However, we have (see the Lemma 6.2 (e'))

$$(6.4) \quad \|(\lambda - A^m)^{-1}\|_{\mathcal{L}(h_-^{-m,n}, h_-^{m,-n})} = O(1).$$

Now, for $\lambda \in \text{Vert}_n^m((2n-1)^m)$ with n large enough, we find that the following decomposition of the resolvent of

$$T(w) = A^m + B(v_0) + B(v_1 + \tilde{w})$$

converges in the space $\mathcal{L}(h_-^{-m})$,

$$(6.5) \quad (\lambda - A^m - B(v_0) - B(v_1 + \tilde{w}))^{-1} = T_\lambda^{-1} + T_\lambda^{-1} K_\lambda (B(v_1 + \tilde{w})T_\lambda^{-1}) + T_\lambda^{-1} K_\lambda (B(v_1 + \tilde{w})T_\lambda^{-1})^2,$$

where

$$T_\lambda = \lambda - A^m - B(v_0),$$

and

$$K_\lambda := \sum_{l \geq 0} (B(v_1 + \tilde{w})T_\lambda^{-1})^{2l}$$

is considered as an element in $\mathcal{L}(h_-^{-m,n})$. Using the Convolution Lemma (c') and the Lemma 6.2 (a'), (b') we can find $n_0 \in \mathbb{N}$ such that, for any $n \geq n_0$ and $\lambda \in \text{Vert}_n^m((2n-1)^m)$, the operator T_λ is invertible in the spaces $\mathcal{L}(h_-^{-m})$ and $\mathcal{L}(h_-^{-m,n})$ in the form (6.2). Using the Convolution Lemma (a') and the Lemma 6.2 (e'), one can obtain

$$\|B(v_1 + \tilde{w})T_\lambda^{-1}\|_{\mathcal{L}(h_-^{-m,n}, h_-^{m,-n})} \leq C_m \|v_1 + \tilde{w}\|_{h_+^{-m}} \|T_\lambda^{-1}\|_{\mathcal{L}(h_-^{-m,n}, h_-^{m,-n})} = O(\varepsilon).$$

Therefore, if $\varepsilon > 0$ is small enough, the sum

$$K_\lambda = \sum_{l \geq 0} (B(v_1 + \tilde{w})T_\lambda^{-1})^{2l}$$

converges in $\mathcal{L}(h_-^{-m,n})$. Then the representation (6.5) follows because

$$B(v_1 + \tilde{w})T_\lambda^{-1} \in \mathcal{L}(h_-^{-m}, h_-^{m,n})$$

by the Convolution Lemma (a') and the Lemma 6.2 (d'), and

$$T_\lambda^{-1} \in \mathcal{L}(h_-^{-m,n}, h_-^{-m})$$

by the Lemma 6.2 (c').

Hence, for $\varepsilon > 0$, $M \geq 1$, and $n_0 \in \mathbb{N}$ as above, the inclusion (6.1) holds. Let remark, that in fact, we have proved the inclusion

$$(6.6) \quad \text{Ext}_M \cup \bigcup_{n \geq n_0} \text{Vert}_n^m((2n-1)^m) \subseteq \text{Resol}(T(w(s))),$$

where $\text{Resol}(T(w(s)))$ denotes the resolvent set of the operator

$$T(w(s)) = A^m + B(v_0) + sB(v_1 + \tilde{w}) \quad \text{for } 0 \leq s \leq 1.$$

Hence, for any contour

$$\Gamma \subset \text{Ext}_M \cup \bigcup_{n \geq n_0} \text{Vert}_n^m((2n-1)^m),$$

and any $0 \leq s \leq 1$, the Riesz projector

$$P(s) := \frac{1}{2\pi i} \int_\Gamma (\lambda - A^m - B(v_0) - sB(v_1 + \tilde{w}))^{-1} d\lambda \in \mathcal{L}(h_-^{-m}),$$

is well defined and depends continuously on s . Since projectors whose difference has a small norm have isomorphic ranges a continuity of the map

$$s \mapsto P(s)$$

implies that the dimension of the range $P(s)$ is independent of s . Therefore the number of eigenvalues of the operators

$$A^m + B(v_0)$$

and

$$A^m + B(v_0) + B(v_1 + \tilde{w})$$

are the same (counted with their algebraic multiplicity) inside Γ . To complete the proof of Theorem 6.3 it is sufficient to apply Theorem 3.5 to the operator

$$A^m + B(v_0)$$

with

$$v_0 \in h^m \subseteq h^0.$$

□

Theorem 6.4. *Let v in h_+^{-m} , and $R \geq 0$. For any $w \in h_+^{-m}$ with*

$$\|w - v\|_{h_+^m} \leq R$$

the spectrum $\text{spec}(A^m + B(w))$ of the operator $T(w) = A^m + B(w)$ consists of a sequence $(\lambda_k(m, w))_{k \geq 1}$ of eigenvalues and the following uniform in w asymptotic formulae

$$\begin{aligned} \lambda_{2n-1}(m, w) &= (2n-1)^{2m} \pi^{2m} + o(n^m), \quad n \rightarrow \infty, \\ \lambda_{2n}(m, w) &= (2n-1)^{2m} \pi^{2m} + o(n^m), \quad n \rightarrow \infty \end{aligned}$$

are hold.

Proof. Let $v \in h_+^{-m}$. Since the set h_+^m is dense in the space h_+^{-m} , one decomposes

$$v = v_0 + v_1,$$

with

$$v_0 \in h_+^m, \quad \text{and} \quad \|v_1\|_{h_+^{-m}} \leq \varepsilon,$$

where $\varepsilon > 0$ will be chosen bellow. We are going to show as above that there exists $n_0 \in \mathbb{N}$ depending on

$$\|v_0\|_{h_+^m}$$

and $R \geq 0$ such that, for any $w = v + w_0 \in h_+^{-m}$ with $\|w_0\|_{h_+^{-m}} \leq R$,

$$(6.7) \quad \bigcup_{n \geq n_0} \text{Vert}_n^m((2n-1)^m) \subseteq \text{Resol}(T(w)),$$

where $\text{Resol}(T(w))$ denotes the resolvent set of the operator

$$T(w) = A^m + B(v_0 + w_0) + B(v_1).$$

Notice, that now we consider the strips $\text{Vert}_n^m(r_n)$ with

$$r_n = \delta(2n-1)^m$$

for some $\delta \in (0, 1]$.

So, let $\lambda \in \text{Vert}_n^m(r_n)$. Using the Convolution Lemma and the Lemma 6.2 (b) one gets

$$\|B(v_0 + w_0)(\lambda - A^m)^{-1}\|_{\mathcal{L}(h_-^{-m,n})} \leq C_m \|v_0 + w_0\|_{h_+^m} \|(\lambda - A^m)^{-1}\|_{\mathcal{L}(h_-^{-m,n})} = \frac{\|(v_0 + w_0)\|_{h_+^m}}{r_n} O(1).$$

Hence, for n large enough and $\lambda \in \text{Vert}_n^m(r_n)$,

$$T_\lambda := \lambda - A^m - B(v_0 + w_0) = (Id - B(v_0 + w_0)(\lambda - A^m)^{-1})(\lambda - A^m)$$

is invertible in $\mathcal{L}(h_-^{-m})$ with inverse

$$(6.8) \quad T_\lambda^{-1} = (\lambda - A^m)^{-1} (Id - B(v_0 + w_0)(\lambda - A^m)^{-1})^{-1}.$$

Further, for n large enough, we can show that the following representation of resolvent of the operator

$$T(w) = A^m + B(v_0 + w_0) + B(v_1)$$

converges in $\mathcal{L}(h_-^{-m})$,

$$(6.9) \quad (\lambda - A^m - B(v_0 + w_0) - B(v_1))^{-1} = T_\lambda^{-1} + T_\lambda^{-1} K_\lambda (B(v_1) T_\lambda^{-1}) + T_\lambda^{-1} K_\lambda (B(v_1) T_\lambda^{-1})^2,$$

where

$$T_\lambda = \lambda - A^m - B(v_0 + w_0),$$

and

$$K_\lambda := \sum_{l \geq 0} (B(v_1) T_\lambda^{-1})^{2l}$$

is considered as an element in $\mathcal{L}(h_-^{-m,n})$. Using the Convolution Lemma and the Lemma 6.2 (e), we get

$$\| B(v_1) T_\lambda^{-1} \|_{\mathcal{L}(h_-^{-m,n}, h_-^{-m,-n})} \leq C_m \| v_1 \|_{h_+^{-m}} \| T_\lambda^{-1} \|_{\mathcal{L}(h_-^{-m,n}, h_-^{-m,-n})} = O(\varepsilon).$$

Hence, if $\varepsilon > 0$ is small enough, the sum

$$K_\lambda = \sum_{l \geq 0} (B(v_1) T_\lambda^{-1})^{2l}$$

converges in the space $\mathcal{L}(h_-^{-m,n})$ and the representation (6.9) then follows because

$$B(v_1) T_\lambda^{-1} \in \mathcal{L}(h_-^{-m}, h_-^{-m,n})$$

by the Convolution Lemma and the Lemma 6.2 (d), and

$$T_\lambda^{-1} \in \mathcal{L}(h_-^{-m,n}, h_-^{-m})$$

by the Lemma 6.2 (c).

Consequently, for some $\varepsilon > 0$ and $n_0 \in \mathbb{N}$ the inclusion (6.7) holds for

$$r_n = \delta n^m, \quad \delta \in (0, 1].$$

So, for any contour

$$\Gamma \subset \bigcup_{n \geq n_0} \text{Vert}_n^m((2n-1)^m),$$

and any $0 \leq s \leq 1$, the Riesz projector

$$P(s) := \frac{1}{2\pi i} \int_\Gamma (\lambda - A^m - B(v_0 + w_0) - B(sv_1))^{-1} d\lambda \in \mathcal{L}(h_-^{-m}).$$

is well defined and depends continuously on s . Since projectors whose difference has a small norm have isomorphic ranges continuity of the map

$$s \mapsto P(s)$$

implies that the dimension of the range $P(s)$ is independent of s . Therefore the number of eigenvalues of the operators

$$A^m + B(v_0 + w_0)$$

and

$$A^m + B(v_0 + w_0) + B(v_1)$$

inside Γ (counted with their algebraic multiplicity) are the same. Applying Theorem 3.5 to the operator

$$A^m + B(v_0 + w_0)$$

one gets:

the spectrum $\text{spec}(T(w))$ of the operator

$$T(w) = A^m + B(w)$$

consists of a sequence $(\lambda_k(m, w))_{k \geq 1}$ of complex-valued eigenvalues, and for any $\delta \in (0, 1]$ there exists $n_0 \in \mathbb{N}$ such that the pairs of eigenvalues $\lambda_{2n-1}(m, w)$, $\lambda_{2n}(m, w)$ there are inside a disc around $(2n-1)^{2m} \pi^{2m}$,

$$|\lambda_{2n-1}(m, w) - (2n-1)^{2m} \pi^{2m}| < \delta (2n-1)^m,$$

$$|\lambda_{2n}(m, w) - (2n-1)^{2m} \pi^{2m}| < \delta (2n-1)^m.$$

So, we conclude that the sequence

$$(\lambda_k(m, w))_{k \geq 1}$$

of eigenvalues satisfies the asymptotic formulae

$$\begin{aligned}\lambda_{2n-1}(m, w) &= (2n-1)^{2m} \pi^{2m} + o(n^m), \quad n \rightarrow \infty, \\ \lambda_{2n}(m, w) &= (2n-1)^{2m} \pi^{2m} + o(n^m), \quad n \rightarrow \infty.\end{aligned}$$

The proof is complete. □

7. ACKNOWLEDGEMENT

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